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# On Eigenvalue Problems with Singularity for Second Order Ordinary Differential Equations*

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**Abstract.** We are dealing with some eigenvalue problems with singularity for second order ordinary differential equations. The existence of a countable set of real eigenvalues and eigenfunctions is established. The high order of smoothness of eigenfunctions are investigated.

## 1. Introduction

The eigenvalue problems for ordinary differential equations with homogeneous boundary conditions arising from applied mathematics have been investigated by many authors and interesting results have been obtained [1–7]. The problems in which eigenvalues are involved in the boundary condition(s), arising from mechanics [2], are called in [2] eigenvalue problems with singularity and have been considered there for particular cases. In this paper, we are dealing with such problems. We shall prove the existence of a countable set of eigenvalues and eigenfunctions and the smoothness of eigenfunctions for second order ordinary differential equations; these questions were not investigated in [2]. The high order of smoothness of eigenfunctions will be used when we approximate eigenvalues and eigenfunctions by finite difference or finite element methods (see [1, 4, 7] for eigenvalue problems with homogeneous boundary conditions).

## 2. On the General Weak Eigenvalue Problem

Let

(i) $H$ and $V$ be infinite dimension Hilbert spaces (on $R$),

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(ii) $a(u, v), (u, v) \in V \times V$ be symmetric, continuous on $V$ and $V$-elliptic.

Then $a(u, v)$ can be used as a new inner product in $V$ and the new norm $\|u\|_a = \{a(u, u)\}^{1/2}$ is *equivalent* to the norm $\|u\|_V$. The space $V$ with this new inner product and new norm is a Hilbert space (on $R$) and is denoted by $V_a$.

Denote by $V_H$ the closure of $V$ with respect to the norm in $H$.

Consider the eigenvalue problem in a weak form: *Find a scalar $\lambda$, called eigenvalue, and an element $u \in V$, $u \neq 0$, called eigen element, such that*

$$a(u, v) = \lambda(u, v)_H \quad \forall v \in V. \tag{2.1}$$

Under assumptions (i) and (ii), we have the following:

**Lemma 1.**
(1) *The weak problem (2.1) has countably many eigenvalues which are real, have no finite limit points and can be arranged as*

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m \leq \dots \lambda_m \to +\infty. \tag{2.2}$$

(2) *The corresponding eigen elements $\{u_m \in V\}$ are orthonormal in $H$, and they form an orthonormal base in $V_H$; the elements $\{\lambda_m^{-1/2} u_m\}$ form an orthonormal base in $V_a$.*

*Proof.* The statement of Lemma 1 is slightly different from that of [5, Theorem 3, p. 195] and [6, Theorem 6.2.1]. However, the proof can be done analogously. We proceed as in [5] and notify only those features which are different from [5].

First, we note that the problem (2.1) can have only real eigenvalues.

Now, we prove the existence of $(\lambda_1, u_1)$. Consider as in [5, pp. 195] the functional

$$\varphi(u) = \frac{a(u, u)}{(u, u)_H} = \frac{\|u\|_a^2}{\|u\|_H^2}, \quad u \in V.$$

Let

$$\lambda_1 = \inf\{\varphi(u) : u \in V\}.$$

Since the imbedding $V \hookrightarrow H$ is continuous and the bilinear form $a(.,.)$ is $V$-elliptic, $\lambda_1 \geq c = \text{const} > 0$. As in [5, pp. 195–196], we can prove the existence of a sequence $\{v_n \in V\}$ which is a Cauchy sequence in $H$ such that

$$\|v_n\|_H = 1, \quad \|v_n\|_a^2 \to \lambda_1 \text{ as } n \to \infty, \tag{2.3}$$

$$a(v_n, \eta) - \lambda_1(v_n, \eta)_H \to 0 \quad \forall \eta \in V \text{ as } n \to \infty, \tag{2.4}$$

$$\|v_n - v_{n'}\|_a^2 - \lambda_1\|v_n - v_{n'}\|_H^2 \to 0 \text{ as } n, n' \to \infty. \tag{2.5}$$

Since $\{v_n\}$ is a Cauchy sequence in $H$, (2.5) shows that it is also a Cauchy sequence in $V_a$. Since $V_a$ is a Hilbert space, there exists an element $u \in V_a$, that is, $u \in V$, such that $\|v_n - u\|_a \to 0$, and hence, $\|v_n - u\|_V \to 0$ as $n \to \infty$ because $a(.,.)$ is $V$-elliptic. By (i), $u \in H$. Since the imbedding $V \hookrightarrow H$ is continuous, $\|v_n - u\|_H \to 0$ as $n \to \infty$. Hence, by passing to the limit, (2.4) yields

$$a(u, \eta) = \lambda_1(u, \eta)_H \quad \forall \eta \in V.$$

Moreover, Eq. (2.3) proves that $\|u\|_H = 1$ and $\|u\|_a^2 = \lambda_1$.

So the existence of $\lambda_1$ and $u_1 = u \in V,\ u_1 \neq 0,\ \|u_1\|_H = 1,\ \|u_1\|_a = \sqrt{\lambda_1}$ satisfying (2.1) is established.

The remainder of the proof can be done analogously to that in [5, pp.189–198]. ∎

## 3. First Eigenvalue Problem with Singularity for Second Order Ordinary Differential Equations

### 3.1. Statement of the Problem in a Strong Form

Let $p(x),\ q(x),\ r(x)$ be given functions on [0, 1] and $\sigma_0,\ \sigma_1,\ s_0, s_1$ be given constants satisfying

$$p(x),\ p'(x),\ q(x),\ r(x) \in C^{\mu}[0,1]\,,\quad \mu = \text{integer} \geq 0\,, \tag{3.1}$$

$$0 < c_0 \leq p(x) \leq c_1,\ \ 0 \leq q(x) \leq c_2,\ \ 0 < c_3 \leq r(x) \leq c_4\,, \tag{3.2}$$

$$\sigma_0 \geq 0,\ \sigma_1 \geq 0,\ \sigma_0 + \sigma_1 > 0,\ s_0 > 0,\ s_1 > 0\,, \tag{3.3}$$

$$c_i = \text{const},\ i = 0, 1, 2, 3, 4.$$

Consider the eigenvalue problem in a strong form: *Find a scalar $\lambda$, called eigenvalue, and a function $u(x) \in C^2[0,1]$ not identically equal to zero, called eigenfunction, satisfying*

$$Lu := -(pu')' + qu = \lambda ru\,,\quad 0 < x < 1\,, \tag{3.4}$$

$$l_0 u := -p(0)u'(0) + \sigma_0 u(0) = \lambda s_0 u(0)\,, \tag{3.5}$$

$$l_1 u := \quad p(1)u'(1) + \sigma_1 u(1) = \lambda s_1 u(1)\,. \tag{3.6}$$

We note that in the problem (3.4)–(3.6), the parameter $\lambda$ is also present in the boundary conditions and that is why it is called an eigenvalue problem with singularity [2, pp. 446–447].

### 3.2. Problem in Weak Form

Denote by $H^k(0,1)$ the Sobolev spaces $W_2^{(k)}(0,1)$ (see Sobolev spaces in [6, 8] for instance).

Let

$$V = H^1(0,1),\ H = \{v | v \in L_2(0,1)\,,\ |v(0)| < \infty\,,\ |v(1)| < \infty\}\,.$$

In other words, $H$ is a subset of $L_2(0,1)$ consisting of $v \in L_2(0,1)$ such that traces $v(0)$ and $v(1)$ are well defined and

$$(u,v)_H = \int_0^1 r(x)u(x)v(x)dx + s_0 u(0)v(0) + s_1 u(1)v(1)\,,\quad \|u\|_H = (u,u)_H^{1/2}\,.$$

$V = H^1(0,1)$ is a Hilbert space; it is obvious that $H$ is also a Hilbert space (on $R$) and $V \subset H$.

If $u \in H^1(0,1)$, then $u \in L_2(0,1)$, and by the trace theorem [6, 8], $u(0)$ and $u(1)$ are well defined: $|u(0)| < \infty\,,\ |u(1)| < \infty$. Hence, $V = H^1(0,1) \subset H$.

From [8, pp. 360–361], we deduce the following:

**Lemma 2.** *In $H^1(0,1)$, the norms*

$$\|u\|_{H^1(0,1)} := \{\|u'\|^2_{L_2(0,1)} + \|u\|^2_{L_2(0,1)}\}^{1/2},$$

$$\|u\|_1 := \|u'\|_{L_2(0,1)} + \|u\|_{L_2(0,1)}, \quad \|u\|_2 := \|u'\|_{L_2(0,1)} + |u(0)| + |u(1)|,$$

$$\|u\|_3 := \|u'\|_{L_2(0,1)} + |u(0)|, \quad \|u\|_4 := \|u'\|_{L_2(0,1)} + |u(1)|$$

*are equivalent to one another.*

Let $u \in H^1(0,\ 1)$. From Lemma 2, we have

$$\|u\|_{L_2(0,1)} \leq \|u\|_{H^1(0,1)}, \quad |u(0)| + |u(1)| \leq \|u\|_2 \leq c_5\|u\|_{H^1(0,1)}, \quad c_5 = \text{const}.$$

Then by addition we see that the imbedding $V = H^1(0,1) \hookrightarrow H$ is continuous.

Now, by the imbedding theorem [8, pp. 359, 372], the imbedding $H^1(0,\ 1) \hookrightarrow L_2(0,\ 1)$ is compact, and by Lemma 2, if the set $\{u\}$ is bounded in $H^1(0,\ 1)$, the sets $\{u(0)\}$ and $\{u(1)\}$ are bounded and therefore possess Cauchy subsequences. Hence, the imbedding $V = H^1(0,\ 1) \hookrightarrow H$ is compact.

Consider in $V = H^1(0,\ 1)$ the bilinear form

$$a(u,v) = \int_0^1 [p(x)u'(x)v'(x) + q(x)u(x)v(x)]dx + \sigma_0 u(0)v(0) + \sigma_1 u(1)v(1), \ u,\ v \in V.$$

Then the general weak eigenvalue problem (2.1) takes the form

$$\int_0^1 [p(x)u'(x)v'(x) + q(x)u(x)v(x)]dx + \sigma_0 u(0)v(0) + \sigma_1 u(1)v(1) =$$

$$\lambda\Big[\int_0^1 r(x)u(x)v(x)dx + s_0 u(0)v(0) + s_1 u(1)v(1)\Big] \quad \forall v \in V = H^1(0,1). \tag{3.7}$$

It is obvious that $a(u, v)$ is symmetric. We have

$$|a(u,v)| \leq c_6\Big[\int_0^1 (|u'|.|v'| + |u|.|v|)dx + |u(0)|.|v(0)| + |u(1)|.|v(1)|\Big], \ c_6 = \text{const}.$$

Then, from Lemma 2 and the Cauchy–Schwarz inequality, we have

$$|a(u,v)| \leq c_7\{\|u\|_V\|v\|_V\}, \quad c_7 = \text{const},$$

that is, $a(u, v)$ is continuous on $V = H^1(0,\ 1)$.

Next, we assume under restriction (3.3) that $\sigma_0 > 0$ for fixing the idea. Let $u \in H^1(0,\ 1)$. We have

$$a(u,u) \geq \int_0^1 p(x)[u'(t)]^2 dt + \sigma_0[u(0)]^2 \geq c_8\{\|u'\|^2_{L_2(0,1)} + [u(0)]^2\}$$

$$\geq c_9\{\|u'\|_{L_2(0,1)} + |u(0)|\}^2, \quad c_8, c_9 = \text{const}.$$

Then, by Lemma 2, $a(.,.)$ is $V = H^1(0,\ 1)$-elliptic.

So assumptions (i) and (ii) are verified, and by Lemma 1, we have the following:

**Theorem 1.**

(1) *The problem (3.7) has countably many eigenvalues which are real, have no finite limit points and can be arranged as (2.2).*

(2) *The corresponding eigenfunctions $\{u_m(x) \in V = H^1(0,1)\}$ form an orthonormal base in $V_H$; the functions $\{\lambda_m^{-1/2} u_m\}$ form an orthonormal base in $V_a$.*

Note that

$$\|u_m\|_H = 1\,, \quad \|u_m\|_a = \sqrt{\lambda_m}\,.$$

### 3.3. Results Concerning the Problem in Strong Form

We have the following:

**Theorem 2.**

(1) *Each solution $(\lambda, u(x))$ of the strong problem (3.4)–(3.6) is a solution of the weak problem (3.7).*

(2) *To the solution $(\lambda_1, u_1(x))$ of the weak problem (3.7) corresponds a solution $(\lambda_1, v_1(x))$ of the strong problem (3.4)–(3.6) such that*

$$v_1 \in C^{\mu+2}[0,\,1]\,,\ \|v_1 - u_1\|_{V=H^1(0,\,1)} = 0\,,\ \|v_1\|_{H^{\mu+2}(0,\,1)} \le c_{10}\,\lambda^{[(\mu+1)/2]+1}\,, \tag{3.8}$$

*where $c_{10}$ is a constant independent of $u_1$ and $v_1$.*

For the proof, we consider at first the boundary problem:

$$Lz := -(p(x)z'(x))' + q(x)z(x) = f(x)\,, \quad 0 < x < 1\,, \tag{3.9}$$

$$l_0 z := -p(0)z'(0) + \sigma_0 z(0) = g_0\,, \tag{3.10}$$

$$l_1 z := \ \ p(1)z'(1) + \sigma_1 z(1) = g_1\,, \tag{3.11}$$

where $p(x)$, $p'(x)$, $q(x)$ verify (3.1)–(3.3) and

$$f(x) \in C^\nu[0,\,1]\,, \quad 0 \le \nu = \text{integer} \le \mu\,,\ g_0,\ g_1 \in R\,. \tag{3.12}$$

Then we have

**Lemma 3.** *The boundary problem (3.9)–(3.12) has a unique solution z:*

$$z \in C^{\nu+2}[0,\,1]\,, \tag{3.13}$$

$$\|z\|_{H^1(0,\,1)} \le c_{11}\{\|f\|_{L_2(0,\,1)} + |g_0| + |g_1|\}\,, \tag{3.14}$$

$$\|z\|_{H^{j+2}(0,1)} \le c_{12}\{\|f\|_{L_2(0,\,1)} + |g_0| + |g_1| + \sum_{k=0}^{j} \|f^{(k)}\|_{L_2(0,\,1)}\}\,, \quad j = 0,\,...,\,\nu\,, \tag{3.15}$$

*where $c_{11}$ and $c_{12}$ are constants independent of z.*

*Proof.* For the existence and uniqueness of $z(x)$ satisfying (3.9)–(3.11) and (3.13), see [4, pp. 79–80]. Thereafter, taking the inner product in $L_2(0, 1)$ of (3.9) by $z(x)$ and taking into account (3.10) and (3.11), we have

$$a(z, z) = (F, z)_H, \tag{3.16}$$

where

$$F = F(x) = \begin{cases} f(x)/r(x), & \text{if } 0 < x < 1, \\ g_0/s_0, & \text{if } x = 0, \\ g_1/s_1, & \text{if } x = 1. \end{cases}$$

Then, by (i) and (ii), Equation (3.16) yields

$$\|z\|_V^2 \le c_{13} \|F\|_H \,.\, \|z\|_H \le c_{14} \|F\|_H \,.\, \|z\|_V \,, \quad c_{13}, c_{14} = \text{const}.$$

This implies (3.14).

Since $z$ verifies (3.13), we can differentiate (3.9) successively and with the help of (3.14), we get (3.15) step by step. ∎

*Proof of Theorem 2.* The first part is obvious — it follows from the inner product in $L_2(0, 1)$ of (3.4) by $v(x) \in V = H^1(0, 1)$ and the boundary conditions (3.5) and (3.6).

For the second part, let $(\lambda_1, u_1(x))$ be the solution of the weak problem (3.7). Then

$$u_1 \in V = H^1(0, 1)\,, \quad a(u_1, u_1) = \lambda_1\,, \quad \|u_1\|_H = 1\,.$$

By the trace theorem [6, 8], $u_1(0)$, $u_1(1)$ are well defined, and by the imbedding theorem [8, pp. 359, 372], $u_1$ is equal almost everywhere (a.e.) on $[0, 1]$ to a function $\tilde{u}_1 \in C[0, 1]$.

Consider the auxiliary boundary value problem.

$$Lw_1 := -(p(x)w_1'(x))' + q(x)w_1(x) = \lambda_1 r(x)\tilde{u}_1(x)\,, \quad 0 < x < 1\,, \tag{3.17}$$

$$l_0 w_1 := -p(0)w_1'(0) + \sigma_0 w_1(0) = \lambda_1 s_0 u_1(0)\,, \tag{3.18}$$

$$l_1 w_1 := p(1)w_1'(1) + \sigma_1 w_1(1) = \lambda_1 s_1 u_1(1)\,. \tag{3.19}$$

By Lemma 3, this problem has a unique solution $w_1$:

$$w_1 \in C^2[0, 1] \tag{3.20}$$

$$\begin{aligned} \|w_1\|_{H^1(0,1)} &\le c_{15}\lambda_1\{\|\tilde{u}_1\|_{L_2(0,1)} + |u_1(0)| + |u_1(1)|\} \\ &= c_{15}\lambda_1\{\|u_1\|_{L_2(0,1)} + |u_1(0)| + |u_1(1)|\} \\ &= c_{15}\lambda_1 \|u_1\|_H \le c_{15}\lambda_1\,, \quad c_{15} = \text{const}. \end{aligned} \tag{3.21}$$

Analogously, Lemma 3 yields

$$\begin{aligned} \|w_1\|_{H^2(0,1)} &\le c_{16}\lambda_1\{\|\tilde{u}_1\|_{L_2(0,1)} + |u_1(0)| + |u_1(1)|\} \\ &= c_{16}\lambda_1\{\|u_1\|_{L_2(0,1)} + |u_1(0)| + |u_1(1)|\} \\ &= c_{16}\lambda_1 \|u_1\|_H \le c_{16}\lambda_1\,, \quad c_{16} = \text{const}. \end{aligned} \tag{3.22}$$

Taking the inner product in $L_2(0,\ 1)$ of two members of (3.17) by $v(x) \in V = H^1(0,\ 1)$, we have after integration by parts

$$\int_0^1 [p(x)w_1'(x)v'(x) + q(x)w_1(x)v(x)]dx + p(0)w_1'(0)v(0) - p(1)w_1'(1)v(1)$$

$$= \lambda_1 \int_0^1 r(x)\tilde{u}_1(x)v(x)dx\,.$$

Then taking into account boundary conditions (3.18) and (3.19), we have

$$\int_0^1 [p(x)w_1'(x)v'(x) + q(x)w_1(x)v(x)]dx + \sigma_0 w_1(0)v(0) + \sigma_1 w_1(1)v(1)$$

$$= \lambda_1 [\int_0^1 r(x)\tilde{u}_1(x)v(x)dx + s_0 u_1(0)v(0) + s_1 u_1(1)v(1)]\,. \qquad (3.23)$$

From (3.7), where $u$ is replaced by $u_1$, and (3.23), where $\tilde{u}_1 = u_1$ a.e. on [0, 1], we obtain

$$a(u_1, v) = a(w_1, v)\,, \quad \forall v \in V.$$

So we have $\|w_1 - u_1\|_a = 0$, and hence,

$$\|w_1 - u_1\|_{H^1(0,\ 1)} = 0\,. \qquad (3.24)$$

From (3.24) and Lemma 2, we have

$$w_1(0) = u_1(0)\,, \quad w_1(1) = u_1(1)\,.$$

From (3.24), we also have $u_1 = w_1$ a.e. on [0, 1]. Since $u_1 = \tilde{u}_1$ a.e. on [0, 1], then $\tilde{u}_1 = w_1$ a.e. on [0,1]. Hence, $\tilde{u}_1 = w_1$ everywhere on [0,1] because they are both continuous on [0,1]. We also deduce that $u_1(0) = \tilde{u}_1(0)$, $u_1(1) = \tilde{u}_1(1)$. Therefore, Eqs. (3.17)–(3.19) coincide with Eqs. (3.4)–(3.6). So $\lambda_1$ and $w_1$ satisfy the problem (3.4)–(3.6) and the relations (3.20)–(3.22) and (3.24) prove that (3.8) is verified for $\mu = 0$ with $v_1 = w_1$.

If $\mu > 0$, we put $K = [(\mu+1)/2]$ and consider auxiliary problems:

$$Lw_{k+1} = \lambda_1 r(x) w_k(x)\,,\ 0 < x < 1\,,\ \ l_0 w_{k+1} = \lambda_1 s_0 w_k(0)\,,\ \ l_1 w_{k+1} = \lambda_1 s_1 w_k(1)\,,$$

$$k = 1, 2, ..., K.$$

Using Lemma 3, we can prove step by step that $w_k$, $k = 2, ..., K+1$, exist and

$$w_k \in C^{2k}[0,\ 1]\,,\ k = 2, ..., K\,,$$

$$w_{K+1} \in \begin{cases} C^{2K+1}[0,\ 1] & \text{if } \mu \text{ is odd} \\ C^{2K+2}[0,\ 1] & \text{if } \mu \text{ is even} \end{cases}, \quad \text{that is, } w_{K+1} \in C^{\mu+2}[0,\ 1]\,,$$

$$\|w_{K+1}\|_{H^{\mu+2}(0,\ 1)} \le c_{17}\lambda_1^{K+1} = c_{17}\lambda_1^{[(\mu+1)/2]+1}\,, \quad c_{17} = \text{const}\,.$$

Now, by subtraction, we have for $k = 1, 2, ..., K$ and $w_0 = u_1$

$$L(w_{k+1} - w_k) = \lambda_1 r(x)[w_k(x) - w_{k-1}(x)], \quad 0 < x < 1,$$
$$l_0(w_{k+1} - w_k) = \lambda_1 s_0[w_k(0) - w_{k-1}(0)],$$
$$l_1(w_{k+1} - w_k) = \lambda_1 s_1[w_k(1) - w_{k-1}(1)].$$

By Lemma 3, we have

$$\|w_{K+1} - w_K\|_{H^1(0,1)} \leq c_{18}\|w_K - w_{K-1}\|_{H^1(0,1)} \leq \cdots$$
$$\leq c_{19}\|w_1 - w_0\|_{H^1(0,1)} = c_{19}\|w_1 - u_1\|_{H^1(0,1)} = 0, \quad c_{18}, \ c_{19} = \text{const}.$$

Therefore,

$$\|w_{K+1} - u_1\|_{H^1(0,1)} \leq \|w_{K+1} - w_K\|_{H^1(0,1)} + \|w_K - w_{K-1}\|_{H^1(0,1)} + \cdots$$
$$+ \|w_1 - w_0\|_{H^1(0,1)} = 0.$$

So (3.8) is verified with $v_1 = w_{K+1}$. ■

### 3.4. On Finite Element Approximation

From the above results, to approximate the first solution of the strong problem (3.4)–(3.6) under assumptions (3.1)–(3.3), we only have to approximate the solution $(\lambda_1, u_1(x))$ of the weak problem (3.7).

For this problem, we can apply the finite element method [7].

## 4. Other Problems with Singularity

We can consider problems with other boundary conditions, for instance, when the boundary condition (3.5) is replaced by one of the following:

$$w(0) = 0, \tag{4.1}$$

$$-p(0)w'(0) + \sigma_0 w(0) = 0, \tag{4.2}$$

or when the boundary condition (3.6) is replaced by one of the following:

$$w(1) = 0, \tag{4.3}$$

$$-p(1)w'(1) + \sigma_1 w(1) = 0. \tag{4.4}$$

These problems are presented in [2] as problems arising from mechanics.

Concerning the problem (3.4), (3.6) and (4.1), we put

$$V = \{v | v \in H^1(0, 1), v(0) = 0\}, \quad H = \{v | v \in L_2(0, 1), |v(1)| < \infty\}. \tag{4.5}$$

The corresponding weak eigenvalue problem is

$$\int_0^1 [p(x)u'(x)v'(x) + q(x)u(x)v(x)]dx + \sigma_1 u(1)v(1)$$

$$= \lambda_1 [\int_0^1 r(x)u(x)v(x)dx + s_1 u(1)v(1)], \quad \forall v \in V. \tag{4.6}$$

Concerning the problem (3.4), (3.6) and (4.2), we put

$$V = H^1(0,1), \quad H = \{v | v \in L_2(0,1), \ |v(0)| < \infty, \ |v(1)| < \infty\}. \tag{4.7}$$

Then the corresponding weak eigenvalue problem is

$$\int_0^1 [p(x)u'(x)v'(x) + q(x)u(x)v(x)]dx + \sigma_0 u(0)v(0) + \sigma_1 u(1)v(1)$$

$$= \lambda_1 [\int_0^1 r(x)u(x)v(x)dx + s_1 u(1)v(1)], \quad \forall v \in V. \tag{4.8}$$

Concerning the problem (3.4), (3.5), and (4.3), we put

$$V = \{v | v \in H^1(0,1), v(1) = 0\}, \quad H = \{v | v \in L_2(0,1), \ |v(0)| < \infty\}. \tag{4.9}$$

The corresponding weak eigenvalue problem is

$$\int_0^1 [p(x)u'(x)v'(x) + q(x)u(x)v(x)]dx + \sigma_0 u(0)v(0)$$

$$= \lambda_1 [\int_0^1 r(x)u(x)v(x)dx + s_0 u(0)v(0)], \quad \forall v \in V. \tag{4.10}$$

Concerning the problem (3.4), (3.5), and (4.4), we put

$$V = H^1(0,1), \quad H = \{v | v \in L_2(0,1), \ |v(0)| < \infty, \ |v(1)| < \infty\}. \tag{4.11}$$

Then the corresponding weak eigenvalue problem is

$$\int_0^1 [p(x)u'(x)v'(x) + q(x)u(x)v(x)]dx + \sigma_0 u(0)v(0) + \sigma_1 u(1)v(1)$$

$$= \lambda_1 [\int_0^1 r(x)u(x)v(x)dx + s_0 u(0)v(0)], \quad \forall v \in V. \tag{4.12}$$

By the same method, we obtain results similar to Theorems 1 and 2, respectively, for eigenvalue problems in weak form (4.6), (4.8), (4.10), and (4.12), and in strong form (3.4), (4.1), (3.6); (3.4), (4.2), (3.6); (3.4), (3.5), (4.3); (3.4), (3.5), (4.4). Note that the spaces $V$ and $H$ in these weak problems are defined by (4.5), (4.7), (4.9), and (4.11), respectively.

As in Subsec. 3.4, we can also apply the finite element method to approximate the solution.

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