On Weak Injectivity of Direct Sums of Modules

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Abstract. Generalizing a notion defined by Jain and López-Permouth [12], we call a module \( Q \in \sigma[M] \) weakly injective (resp. weakly tight) in \( \sigma[M] \) if, for every finitely generated submodule \( N \) of the \( M \)-injective hull \( \mathcal{Q} \), \( N \) is contained in a submodule \( Y \) of \( \mathcal{Q} \) such that \( Y \cong Q \) (resp. \( N \) is finitely \( Q \)-cogenerated). For some classes \( M \) of weakly injectives in \( \sigma[M] \), we study the instances in which direct sums of modules from \( M \) are weakly injective in \( \sigma[M] \). In particular, we get necessary and sufficient conditions for \( \sum \)-weak injectivity or \( \sum \)-weak tightness of the injective hull of a simple module. As a consequence, we get characterizations for q.f.d. rings by means of weakly injective modules given by Al-Huzali, Jain, and López-Permouth [2].

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary. For a module \( M \) over a ring \( R \), we write \( M_R(RM) \) to indicate that \( M \) is a right (left) \( R \)-module. We denote the category of all right \( R \)-modules by \( \text{Mod-}R \), and for any \( M \in \text{Mod-}R \), \( \text{of}M \) stands for the full subcategory of \( \text{Mod-}R \) whose objects are submodules of \( M \)-generated modules (see [16]). Given a module \( X_R \), the injective hull of \( X \) in \( \text{Mod-}R \) (resp. in \( \sigma[M] \)) is denoted by \( E(X) \) (resp. \( \mathcal{X} \)).

Let \( M_R \) be a fixed module and \( K \) a class of simple modules in \( \sigma[M] \). We denote \( \text{Soc}_K(X) = \sum \{ A \subseteq X | A \cong P \text{ for some } P \in K \} \).

Recall in [7] that \( X \in \sigma[M] \) is said to be countably thick relative to \( K \) if \( \text{Soc}_K(X/A) \) is finitely generated for all \( A \subseteq X \).

We consider some cases.

Case 1. If \( K \) is the class of all simple modules in \( \sigma[M] \), then \( X \in \sigma[M] \) is countably thick relative to \( K \) if and only if all factor modules of \( X \) have a finite uniform dimension, that is, \( X \) is q.f.d. (see [8, 9]).
Case 2. Modules in \(\sigma[M]\) which are countably thick relative to \(\{P\}\) for all simple \(P\) in \(\sigma[M]\), coincide with countably distributive modules in \(\sigma[M]\) (see Lemma 1).

A module \(X_R\) is called countably distributive \([4, 5]\) if

\[
A + \bigcap_{i \in \mathbb{N}} A_i = \bigcap_{i \in \mathbb{N}} \left( A + \bigcap_{j \in \mathbb{N} \setminus \{i\}} A_j \right)
\]

for all submodules \(A\) of \(X\) and all families \(\{A_i\}_{i \in \mathbb{N}}\) of submodules of \(X\).

For a module \(X_R\) and a module property \(P\), \(X\) is said to be \(\sum P\) in cases where every direct sum of copies of \(X\) enjoys the property \(P\) (see [1]). Also, we call \(X\) locally \(P\) in cases where every finitely generated submodule of \(X\) enjoys the property \(P\).

\(X\) is said to be a CFD module if every cyclic submodule of \(X\) has a finite uniform dimension (see [3]). It is easy to see that every locally Noetherian module is locally q.f.d. and every locally q.f.d. module is CFD.

The paper contains further developments of the ideas in [2, 4, 5, 8].

2. Thickness Relative to \(K\)

With the above definitions, we have the following necessary lemma [5, Theorem 1].

**Lemma 1.** For a module \(X \in \sigma[M]\), the following conditions are equivalent:

(a) \(X\) is countably distributive;
(b) for each simple \(P \in \sigma[M]\), \(X\) is countably thick relative to \(\{P\}\);
(c) in each factor module of \(X\), any independent system \(A\) of nonzero isomorphic submodules is finite;
(d) \(\sum (a_i : \sum_{j \in \mathbb{N} \setminus \{i\}} a_j R) = R\) for every system \(\{a_i\}_{i \in \mathbb{N}}\) of elements of \(X\), where \(a_i : B = \{r \in R \mid ar \in B\}\).

The next proposition is a generalization of [8, Theorem], [14, Proposition 2.2], and [5, Theorem 2].

**Proposition 2.** For a module \(X \in \sigma[M]\) and any class \(K\) of simple modules in \(\sigma[M]\), the following conditions are equivalent:

(a) \(X\) is countably thick relative to \(K\);
(b) for any submodule \(A\) of \(X\) and for every properly ascending chain \(B_1 \subset B_2 \subset \cdots\) of submodules of \(X/A\), there exists \(n \in \mathbb{N}\) such that \(\text{Soc}_K(B_n) = \text{Soc}_K(B_m)\) for all \(m \geq n\);
(c) for each submodule \(K\) of \(X\), there exists a finitely generated submodule \(T\) of \(K\) such that \(\text{Hom}_R(K/T, P) = 0\) for all \(P \in K\).

**Proof.** (a) \(\Rightarrow\) (b). Let \(B_1 \subset B_2 \subset \cdots\) be a properly ascending chain of submodules of the factor module \(X/A\). Since \(X\) is countably thick relative to \(K\), \(\text{Soc}_K(\bigcup_{i \in \mathbb{N}} B_i)\) is finitely generated. So \(\text{Soc}_K(\bigcup_{i \in \mathbb{N}} B_i) = \text{Soc}_K(B_n)\) for some \(n\). Hence, (b) follows.

(b) \(\Rightarrow\) (c). Suppose for \(Z_R\), every finitely generated submodule \(T\) of \(Z\) is contained in a maximal submodule \(Q\) of \(Z\), for which \(Z/Q \simeq P \in K\). Then by induction it is easy to see for each \(n \in \mathbb{N}\) the existence of maximal submodules \(Q_1, \ldots, Q_n\) of \(Z\) and elements \(x_1, \ldots, x_n \in Z\), for which \(x_i \notin Q_i, x_i \in Q_j\) for all \(j > i\) and \(Z/Q_i \simeq P_i \in K\).
where \( i \leq n \). (At the \( n \)th step of our process, we choose a maximal submodule \( Q_n \) and an arbitrary element \( x_n \notin Q_n \) such that \( \sum_{i=1}^{n-1} x_i R \subseteq Q_n \) and \( Z/Q_n \cong P_n \in \mathcal{K} \).) Let

\[
Y = Z \bigcap_{i \in \mathbb{N}} Q_i,
\]

\[
Y_n = \left( \bigcap_{i=1}^{n} Q_i \right) / \left( \bigcap_{i \in \mathbb{N}} Q_i \right),
\]

and

\[
Z_n = \left( \bigcap_{i=n+1}^{\infty} Q_i \right) / \left( \bigcap_{i \in \mathbb{N}} Q_i \right),
\]

for all \( n \in \mathbb{N} \). We observe that \( Y = Y_n \oplus Z_n \) (see [9, p.43]). But then the properly ascending chain \( Z_1 \subset Z_2 \subset \cdots \) of submodules of \( Y \) contradicts condition (b).

(c) \( \Rightarrow \) (a). Suppose for a system \((P_i)_{i \in \mathbb{N}}\) of simple modules from \( \mathcal{K} \) and submodules \( L \subseteq K \subseteq X \), we have \( K/L \cong \bigoplus_{i \in \mathbb{N}} P_i \). Choose a finitely generated submodule \( T \) of \( K \) for which \( \text{Hom}_R(K/T, P_i) = 0 \) for all \( i \in \mathbb{N} \). Then \( \text{Hom}_R(K/(T + L), P_i) = 0 \) for all \( i \in \mathbb{N} \). But \( K/(T + L) \) is a homomorphic image of \( \bigoplus_{i=1}^{\infty} P_i \), so we have \( K/(T + L) = 0 \), that is, \( T + L = K \) and then \( K/L \cong T/(T \cap L) \), which contradicts the fact that \( T \) is finitely generated.

The next lemma is a generalization of [5, Corollary 6] and [6, Lemma 7]. Recall that a class \( \mathcal{K} \subseteq \sigma[M] \) is called a Serre class in \( \sigma[M] \) if it is closed under submodules, factor modules and extensions in \( \sigma[M] \) (e.g., [10]).

**Proposition 3.** The class of all modules in \( \sigma[M] \) which are countably thick relative to \( \mathcal{K} \) is a Serre class in \( \sigma[M] \).

**Proof.** Let \( A \) be a submodule of \( X_R \). It is clear that if \( X \) is countably thick relative to \( \mathcal{K} \), then \( A \) and \( X/A \) are countably thick relative to \( \mathcal{K} \). Moreover, suppose \( A \) and \( X/A \) are countably thick relative to \( \mathcal{K} \). By Proposition 2, for a submodule \( K \subseteq X \), we choose finitely generated submodules \( T_1 \subseteq K \cap A \) and \( T_2 \subseteq K \) for which \( \text{Hom}_R((K \cap A)/T_1, P) = 0 \) and \( \text{Hom}_R(K/(T_2 + K \cap A), P) = 0 \) for all \( P \in \mathcal{K} \). Let \( i : (K \cap A)/T_1 \to K/T_1 \) be the inclusion map and \( \pi : K/T_1 \to K/(T_1 + T_2) \) and \( \tau : K/(T_1 + T_2) \to K/(K \cap A + T_2) \) the natural projection maps. Then for \( f \in \text{Hom}_R(K/(T_1 + T_2), P) \), we find that \( f \pi i = 0 \) and \( f = g \tau \) for some \( g : K/(T_2 + K \cap A) \to P \). Since \( g = 0 \), then \( f = 0 \). By Proposition 2, we conclude that \( X \) is countably thick relative to \( \mathcal{K} \).

The next two propositions are similar to [16, 27.2 and 27.3]. We present the proofs for the convenience of the readers.

**Proposition 4.** Let \( K \) be a class of simple modules in \( \sigma[M] \).

1. Let \( 0 \to N' \to N \to N'' \to 0 \) be an exact sequence in \( \sigma[M] \).
(i) If $N$ is locally countably thick relative to $\mathcal{K}$, then $N'$ and $N''$ are locally countably thick relative to $\mathcal{K}$.

(ii) If $N'$ is countably thick relative to $\mathcal{K}$ and $N''$ is locally countably thick relative to $\mathcal{K}$, then $N$ is locally countably thick relative to $\mathcal{K}$.

(2) The direct sum of modules in $\sigma[M]$ which are locally countably thick relative to $\mathcal{K}$ is again locally countably thick relative to $\mathcal{K}$.

Proof. (1)(i) The proof is straightforward.

(1)(ii) Let $N'$ be countably thick relative to $\mathcal{K}$, $N''$ locally countably thick relative to $\mathcal{K}$, and $K$ a finitely generated submodule of $N$. Then by using the exact commutative diagram

$$
\begin{array}{c}
0 \rightarrow K \cap N' \rightarrow K \rightarrow K/K \cap N' \rightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0
\end{array}
$$

we see that $K \cap N'$ and $K/K \cap N'$ are countably thick relative to $\mathcal{K}$. By Proposition 3, $K$ is also countably thick relative to $\mathcal{K}$.

(2) By Proposition 3, every finite direct sum of countably thick relative to $\mathcal{K}$ modules is countably thick relative to $\mathcal{K}$. If $N, X \in \sigma[M]$ are locally countably thick relative to $\mathcal{K}$, then $N \oplus X$ is locally countably thick relative to $\mathcal{K}$. Let $K$ be a submodule of $N \oplus X$ generated by the elements $(n_1, x_1), \ldots, (n_r, x_r)$ in $K$ (with $n_i \in N, x_i \in X, r \in \mathbb{N}$). The submodules $N' = \sum_{i \leq r} n_i R \subseteq N$ and $X' = \sum_{i \leq r} x_i R \subseteq X$ are countably thick relative to $\mathcal{K}$ by assumption, and hence, $N' \oplus X'$ is countably thick relative to $\mathcal{K}$. Since $K \subseteq N' \oplus X'$, $K$ is also countably thick relative to $\mathcal{K}$.

By induction, we see that every finite direct sum of locally countably thick relative to $\mathcal{K}$ modules in $\sigma[M]$ is locally countably thick relative to $\mathcal{K}$. Then the corresponding assertion is true for arbitrary sums since every finitely generated submodule of it is contained in a finite partial sum.

Corollary 5. For a module $M_R$ and any class $\mathcal{K}$ of simple modules in $\sigma[M]$, the following conditions are equivalent:

(a) $M$ is locally countably thick relative to $\mathcal{K}$;
(b) every cyclic submodule of $M$ is countably thick relative to $\mathcal{K}$;
(c) $M^{(\mathcal{R})}$ is locally countably thick relative to $\mathcal{K}$;
(d) $\sigma[M]$ has a set of generators consisting of modules which are countably thick relative to $\mathcal{K}$;
(e) every finitely generated (cyclic) module in $\sigma[M]$ is countably thick relative to $\mathcal{K}$;
(f) every module in $\sigma[M]$ is locally countably thick relative to $\mathcal{K}$.

Proof. This follows Proposition 4 and the fact that the finitely generated submodules of $M^{(\mathcal{R})}$ form a set of generators of $\sigma[M]$.

3. Thickness vs. Weak Injectivity

Given a module $M_R$ and $Q \in \sigma[M]$, we say that $Q$ is weakly injective in $\sigma[M]$ if, for every finitely generated submodule $N$ of $\hat{Q}$, there exists a submodule $Y$ of $\hat{Q}$ such that
N \subseteq Y \simeq Q. Note that this notion is different from weakly $M$-injective as defined in [16,16.9].

We say that $Q$ is tight in $\sigma[M]$ if every finitely generated submodule $N$ of $\tilde{Q}$ is embeddable in $Q$, and $Q$ is weakly tight in $\sigma[M]$ if every finitely generated submodule $N$ of $\tilde{Q}$ is embeddable in a direct sum of copies of $Q$.

A module $Q_R$ is said to be weakly injective [12] (tight [11]) if it is weakly injective (tight) in $\sigma[R] = \text{Mod} - R$. It is clear that every weakly injective in $\sigma[M]$ is tight in $\sigma[M]$, and every tight module in $\sigma[M]$ is weakly tight in $\sigma[M]$, but weak tightness does not imply tightness. For this, consider the category $\sigma[Q/Z]$, i.e., the torsion $Z$-modules. Then $X = Q/Z \oplus Z/pZ$ is a cogenerator in $\sigma[Q/Z]$ and $X \simeq Q/Z \oplus Z/p\infty$. Obviously, $X$ is weakly tight in $\sigma[Q/Z]$ (since the category $\sigma[Q/Z]$ is locally artinian). But $Z/p^2Z \oplus Z/p^2Z$ is a finitely generated submodule of $X$ which is not embeddable in $X$.

By the proof of [15, Lemma 2], we obtain the following:

**Lemma 6.** For a module $M_R$ and $Q$, $L \in \sigma[M]$, we have
(a) if $Q$ and $L$ are weakly injective in $\sigma[M]$, then $Q \oplus L$ is also weakly injective in $\sigma[M]$;
(b) if $Q$ is an essential submodule of $L$ and $Q$ is weakly injective in $\sigma[M]$, then $L$ is weakly injective in $\sigma[M]$.

Now, we are in the position to prove the main result.

**Theorem 7.** For a module $M_R$ and any class $K$ of simples in $\sigma[M]$, the following conditions are equivalent:
(a) $M$ is locally countably thick relative to $K$;
(b) every direct sum $\bigoplus_\lambda E_\lambda$ of injectives in $\sigma[M]$, where each $E_\lambda$ is essential over $\text{Soc}_K(E_\lambda)$, is weakly injective in $\sigma[M]$;
(c) every direct sum $\bigoplus_\lambda M_\lambda$ of weakly injectives in $\sigma[M]$, where each $M_\lambda$ is essential over $\text{Soc}_K(E_\lambda)$, is weakly injective in $\sigma[M]$;
(d) every direct sum $\bigoplus_\lambda P_\lambda$, where $P_\lambda \in K$, is weakly injective in $\sigma[M]$;
(e) every direct sum $\bigoplus_\lambda P_\lambda$, where $P_\lambda \in K$, is weakly tight in $\sigma[M]$.

**Proof.** (a) $\Rightarrow$ (b). Consider $X = \bigoplus_\lambda E_\lambda$, where $E_\lambda$ is injective in $\sigma[M]$ for every $\lambda \in \Lambda$ and $\text{Soc}_K(E_\lambda)$ is essential in $E_\lambda$.

Let $N$ be a finitely generated submodule of $\tilde{X}$. By the hypothesis, $\text{Soc}_K(N)$ is finitely generated, that is,

$$\text{Soc}_K(N) = P_1 \oplus \cdots \oplus P_n$$

with $P_i \simeq P_i'$ for some $P_i' \in K$ ($1 \leq i \leq n$).

So

$$\text{Soc}_K(N) \subseteq \text{Soc}_K(\tilde{X}) = \text{Soc}_K(X) \subseteq X,$$

and hence, $\text{Soc}_K(N) \subseteq E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_m}$ for some finite $\{\lambda_1, \ldots, \lambda_m\}$. This implies that $E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_m}$ contains an injective hull $E$ of $\text{Soc}_K(N)$. Since $E$ is injective and contained in $X$, we may write $X = E \oplus K$ for some submodule $K$ of $X$. On the other hand, let $\tilde{N}$ be an injective hull of $N$ inside $\tilde{X}$. Then $\tilde{N} = \bigoplus_{i=1}^n P_{i\lambda} \simeq E$. Since $\text{Soc}_K(N)$ is essential in $\tilde{N}$, it follows that $\tilde{N} \cap K = 0$. So let $Y = \tilde{N} \oplus K \simeq E \oplus K = X$, proving that $X$ is weakly injective in $\sigma[M]$.
(b) ⇒ (c). Consider the module $X = \bigoplus_{\lambda} M_{\lambda}$, a direct sum of weakly injectives in $\sigma[M]$, where each $M_{\lambda}$ is essential over $\text{Soc}_{\mathcal{K}}(M)$. Let $N$ be a finitely generated submodule of $X$. By (b), the direct sum $\bigoplus_{\lambda} M_{\lambda}$ is weakly injective in $\sigma[M]$ and $X \cong \bigoplus_{\lambda} M_{\lambda} \subseteq \tilde{X}$. Hence, by (b), there exists a submodule $Y \subseteq \tilde{X}$ such that $N \subseteq Y$ and $Y \cong \bigoplus_{\lambda} M_{\lambda}$.

Write $Y = \bigoplus_{\lambda} Y_{\lambda}$, such that $Y_{\lambda} \cong M_{\lambda}$ for all $\lambda \in \Lambda$. Since $N$ is finitely generated, there exists a finite subset $\Gamma \subseteq \Lambda$ such that $N \subseteq \bigoplus_{\lambda \in \Gamma} Y_{\lambda} = \bigoplus_{\lambda \in \Gamma} Y_{\lambda}$. Since the $M_{\lambda}$’s are weakly injective in $\sigma[M]$, the finite sum $\bigoplus_{\lambda \in \Gamma} Y_{\lambda}$ is weakly injective in $\sigma[M]$, and therefore, there exists $X_{1} \cong \bigoplus_{\lambda \in \Gamma} Y_{\lambda} \cong \bigoplus_{\lambda \in \Gamma} M_{\lambda}$ such that $N \subseteq X_{1} \subseteq \bigoplus_{\lambda \in \Gamma} Y_{\lambda}$. But then $N \subseteq X_{1} \oplus (\bigoplus_{\lambda \notin \Gamma} Y_{\lambda}) \cong X$, proving our claim.

(c) ⇒ (d) and (d) ⇒ (e) are trivial.

(e) ⇒ (a). Let $C$ be a finitely generated submodule of $M$. If $\text{Soc}_{\mathcal{K}}(C) = 0$, we are done. Suppose $0 \neq \text{Soc}_{\mathcal{K}}(C) = \bigoplus_{\lambda} P_{\lambda}$, where $P_{\lambda} \cong P_{\lambda}'$ for some $P_{\lambda}' \in \mathcal{K}$. We show that $\text{Soc}_{\mathcal{K}}(C)$ is finitely generated.

For this, consider the diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \bigoplus_{\lambda} P_{\lambda} \\
\downarrow \varphi & & \gamma \\
\bigoplus_{\lambda} P_{\lambda} & \longrightarrow & C
\end{array}
$$

where $\varphi$ and $\gamma$ are inclusion homomorphisms. By $M$-injectivity, there exists $\psi$ such that $\psi \gamma = \varphi$. By our hypothesis, $\bigoplus_{\lambda} P_{\lambda}$ is weakly tight in $\sigma[M]$, hence, $\text{Im}\varphi \subseteq \text{Im}\psi$ is finitely $\bigoplus_{\lambda} P_{\lambda}$-cogenerated. Therefore, $\text{Soc}_{\mathcal{K}}(C)$ is embeddable in $P_{\lambda_{1}} \oplus \cdots \oplus P_{\lambda_{n}}$, for some $\lambda_{1}, \ldots, \lambda_{n} \in \Lambda$. Since each $P_{\lambda}$ is uniform, $\text{Soc}_{\mathcal{K}}(C)$ has a finite uniform dimension and is therefore finitely generated.

For $\mathcal{K} = \{P\}$, $P$ is a simple module in $\sigma[M]$, and we obtain the following:

**Corollary 8.** For a module $M_{\mathcal{R}}$ and a simple module $P$ in $\sigma[M]$, the following conditions are equivalent:

(a) $M$ is locally countably thick relative to $\{P\}$;
(b) every direct sum $\bigoplus_{\lambda} E_{\lambda}$ of injectives in $\sigma[M]$, where each $E_{\lambda}$ is essential over $\text{Soc}_{\{P\}}(E_{\lambda})$, is weakly injective in $\sigma[M]$;
(c) every direct sum $\bigoplus_{\lambda} M_{\lambda}$ of weakly injectives in $\sigma[M]$, where each $M_{\lambda}$ is essential over $\text{Soc}_{\{P\}}(M_{\lambda})$, is weakly injective in $\sigma[M]$;
(d) $\widetilde{P}$ is $\sum$-weakly injective in $\sigma[M]$;
(e) $\widetilde{P}$ is $\sum$-weakly tight in $\sigma[M]$.

In the case where Corollary 8 applies for all simple modules, we have the following:

**Corollary 9.** For a module $M_{\mathcal{R}}$, the following conditions are equivalent:

(a) $M$ is locally countably distributive;
(b) for each simple module $P$ in $\sigma[M]$, every direct sum $\bigoplus_{\lambda} E_{\lambda}$ of injectives in $\sigma[M]$, where each $E_{\lambda}$ is essential over $\text{Soc}_{\{P\}}(E_{\lambda})$, is weakly injective in $\sigma[M]$;
(c) for each simple module $P$ in $\sigma[M]$, every direct sum $\bigoplus_{\lambda} M_{\lambda}$ of weakly injectives in $\sigma[M]$, where each $M_{\lambda}$ is essential over $\text{Soc}_{\{P\}}(M_{\lambda})$, is weakly injective in $\sigma[M]$;
(d) for each simple module $P$ in $\sigma[M]$, $\widetilde{P}$ is $\sum$-weakly injective in $\sigma[M]$;
(e) for each simple module $P$ in $\sigma[M]$, $\widetilde{P}$ is $\sum$-weakly tight in $\sigma[M]$.  

Finally, taking for K all simple R-modules (in $\sigma[M]$) and using Theorem 7 and arguments from [2, Theorem], we have the following:

**Proposition 10.** For a module $M_R$, the following conditions are equivalent:

(a) $M$ is locally q.f.d.;

(b) every direct sum of injectives in $\sigma[M]$ is weakly injective in $\sigma[M]$;

(c) direct sums of weakly injectives are weakly injective in $\sigma[M]$;

(d) every direct sum $\bigoplus \chi P_\chi$, where each $P_\chi$ is simple in $\sigma[M]$, is weakly injective in $\sigma[M]$;

(e) every direct sum $\bigoplus \chi P_\chi$, where each $P_\chi$ is simple in $\sigma[M]$, is weakly tight in $\sigma[M]$.

**Proof.** (a) $\Rightarrow$ (b). For this, the proof of (1) $\Rightarrow$ (2) of [2, Theorem] applies.

(b) $\Rightarrow$ (c). This is obtained by the proof of (b) $\Rightarrow$ (c) of Theorem 7.

Other implications follow from Theorem 7.

In the case of $M = R_R$, from Proposition 10, we obtain [2, Theorem].

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