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On Weak Injectivity of Direct Sums of Modules

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Abstract. Generalizing a notion defined by Jain and López–Permouth [12], we call a module $Q \in \sigma[M]$ weakly injective (resp. weakly tight) in $\sigma[M]$ if, for every finitely generated submodule N of the M-injective hull $\widehat{Q}$, N is contained in a submodule Y of $\widehat{Q}$ such that $Y \simeq Q$ (resp. N is finitely Q-cogenerated). For some classes $\mathcal{M}$ of weakly injectives in $\sigma[M]$, we study the instances in which direct sums of modules from $\mathcal{M}$ are weakly injective in $\sigma[M]$. In particular, we get necessary and sufficient conditions for $\sum$-weak injectivity or $\sum$-weak tightness of the injective hull of a simple module. As a consequence, we get characterizations for q.f.d. rings by means of weakly injective modules given by Al-Huzali, Jain, and López–Permouth [2].

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary. For a module M over a ring R, we write $M_R\,({}_RM)$ to indicate that M is a right (left) R-module. We denote the category of all right R-modules by Mod-R, and for any $M \in \text{Mod}-R$, $\sigma[M]$ stands for the full subcategory of Mod $- R$ whose objects are submodules of M-generated modules (see [16]). Given a module X_R, the injective hull of X in Mod $- R$ (resp. in $\sigma[M]$) is denoted by $E(X)$ (resp. $\widehat{X}$).

Let M_R be a fixed module and $\mathcal{K}$ a class of simple modules in $\sigma[M]$. We denote

$$\text{Soc}_{\mathcal{K}}(X) = \sum\{A \subseteq X \mid A \simeq P \text{ for some } P \in \mathcal{K}\}.$$

Recall in [7] that $X \in \sigma[M]$ is said to be *countably thick relative to* $\mathcal{K}$ if $\text{Soc}_{\mathcal{K}}(X/A)$ is finitely generated for all $A \subseteq X$.

We consider some cases.

Case 1. If $\mathcal{K}$ is the class of all simple modules in $\sigma[M]$, then $X \in \sigma[M]$ is countably thick relative to $\mathcal{K}$ if and only if all factor modules of X have a finite uniform dimension, that is, X is q.f.d. (see [8, 9]).

Case 2. Modules in $\sigma[M]$ which are countably thick relative to $\{P\}$ for all simple P in $\sigma[M]$, coincide with countably distributive modules in $\sigma[M]$ (see Lemma 1).

A module X_R is called countably distributive [4, 5] if

$$A + \bigcap_{i \in I\!N} A_i = \bigcap_{i \in I\!N} \Big(A + \bigcap_{j \in I\!N \setminus \{i\}} A_j \Big)$$

for all submodules A of X and all families $\{A_i\}_{I\!N}$ of submodules of X.

For a module X_R and a module property $\mathbb{P}$, X is said to be $\sum - \mathbb{P}$ in cases where every direct sum of copies of X enjoys the property $\mathbb{P}$ (see [1]). Also, we call X locally $\mathbb{P}$ in cases where every finitely generated submodule of X enjoys the property $\mathbb{P}$.

X is said to be a *CFD module* if every cyclic submodule of X has a finite uniform dimension (see [3]). It is easy to see that every locally Noetherian module is locally q.f.d. and every locally q.f.d. module is CFD.

The paper contains further developments of the ideas in [2, 4, 5, 8].

2. Thickness Relative to $\mathcal{K}$

With the above definitions, we have the following necessary lemma [5, Theorem 1].

Lemma 1. *For a module $X \in \sigma[M]$, the following conditions are equivalent:*
(a) *X is countably distributive;*
(b) *for each simple $P \in \sigma[M]$, X is countably thick relative to $\{P\}$;*
(c) *in each factor module of X, any independent system A of nonzero isomorphic submodules is finite;*
(d) *$\sum_{i \in I\!N} (a_i : \sum_{j \in I\!N \setminus \{i\}} a_j R) = R$ for every system $\{a_i\}_{I\!N}$ of elements of X, where $(a : B) = \{r \in R \setminus ar \in B\}$.*

The next proposition is a generalization of [8, Theorem], [14, Proposition 2.2], and [5, Theorem 2].

Proposition 2. *For a module $X \in \sigma[M]$ and any class K of simple modules in $\sigma[M]$, the following conditions are equivalent:*
(a) *X is countably thick relative to K;*
(b) *for any submodule A of X and for every properly ascending chain $B_1 \subset B_2 \subset \cdots$ of submodules of X/A, there exists $n \in I\!N$ such that $Soc_{\mathcal{K}}(B_n) = Soc_{\mathcal{K}}(B_m)$ for all $m \geq n$;*
(c) *for each submodule K of X, there exists a finitely generated submodule T of K such that $Hom_R(K/T, P) = 0$ for all $P \in \mathcal{K}$.*

Proof. (a) $\Rightarrow$ (b). Let $B_1 \subset B_2 \subset \cdots$ be a properly ascending chain of submodules of the factor module X/A. Since X is countably thick relative to $\mathcal{K}$, $\mathrm{Soc}_{\mathcal{K}}(\bigcup_{I\!N} B_i)$ is finitely generated. So $\mathrm{Soc}_{\mathcal{K}}(\bigcup_{I\!N} B_i) = \mathrm{Soc}_{\mathcal{K}}(B_n)$ for some n. Hence, (b) follows.

(b) $\Rightarrow$ (c). Suppose for Z_R, every finitely generated submodule T of Z is contained in a maximal submodule Q of Z, for which $Z/Q \simeq P \in \mathcal{K}$. Then by induction it is easy to see for each $n \in I\!N$ the existence of maximal submodules $Q_1, \ldots, Q_n$ of Z and elements $x_1, \ldots, x_n \in Z$, for which $x_i \notin Q_i$, $x_i \in Q_j$ for all $j > i$ and $Z/Q_i \simeq P_i \in \mathcal{K}$, where $i \leq n$. (At the nth step of our process, we choose a maximal submodule Q_n and an arbitrary element $x_n \notin Q_n$ such that $\sum_{i=1}^{n-1} x_i R \subseteq Q_n$ and $Z/Q_n \simeq P_n \in \mathcal{K}$.) Let

$$Y = Z \Big/ \bigcap_{i \in I\!N} Q_i,$$

$$Y_n = \Big(\bigcap_{i=1}^{n} Q_i\Big) \Big/ \Big(\bigcap_{i \in I\!N} Q_i\Big),$$

and

$$Z_n = \Big(\bigcap_{i=n+1}^{\infty} Q_i\Big) \Big/ \Big(\bigcap_{i \in I\!N} Q_i\Big),$$

for all $n \in I\!N$. We observe that $Y = Y_n \oplus Z_n$ (see [9, p. 43]). But then the properly ascending chain $Z_1 \subset Z_2 \subset \cdots$ of submodules of Y contradicts condition (b).

(c) $\Rightarrow$ (a). Suppose for a system $(P_i)_{I\!N}$ of simple modules from $\mathcal{K}$ and submodules $L \subseteq K \subseteq X$, we have $K/L \simeq \bigoplus_{i=1}^{\infty} P_i$. Choose a finitely generated submodule T of K for which $\mathrm{Hom}_R(K/T, P_i) = 0$ for all $i \in I\!N$. Then $\mathrm{Hom}_R(K/(T+L), P_i) = 0$ for all $i \in I\!N$. But $K/(T+L)$ is a homomorphic image of $\bigoplus_{i=1}^{\infty} P_i$, so we have $K/(T+L) = 0$, that is, $T + L = K$ and then $K/L \simeq T/(T \cap L)$, which contradicts the fact that T is finitely generated. ∎

The next lemma is a generalization of [5, Corollary 6] and [6, Lemma 7]. Recall that a class $\mathcal{K} \subseteq \sigma[M]$ is called a Serre class in $\sigma[M]$ if it is closed under submodules, factor modules and extensions in $\sigma[M]$ (e.g., [10]).

Proposition 3. *The class of all modules in $\sigma[M]$ which are countably thick relative to $\mathcal{K}$ is a Serre class in $\sigma[M]$.*

Proof. Let A be a submodule of X_R. It is clear that if X is countably thick relative to $\mathcal{K}$, then A and X/A are countably thick relative to $\mathcal{K}$. Moreover, suppose A and X/A are countably thick relative to $\mathcal{K}$. By Proposition 2, for a submodule $K \subseteq X$, we choose finitely generated submodules $T_1 \subseteq K \cap A$ and $T_2 \subseteq K$ for which $\mathrm{Hom}_R((K \cap A)/T_1, P) = 0$ and $\mathrm{Hom}_R(K/(T_2 + K \cap A), P) = 0$ for all $P \in \mathcal{K}$. Let $i : (K \cap A)/T_1 \to K/T_1$ be the inclusion map and $\pi : K/T_1 \to K/(T_1 + T_2)$ and $\tau : K/(T_1 + T_2) \to K/(K \cap A + T_2)$ the natural projection maps. Then for $f \in \mathrm{Hom}_R(K/(T_1 + T_2), P)$, we find that $f\pi i = 0$ and $f = g\tau$ for some $g : K/(T_2 + K \cap A) \to P$. Since $g = 0$, then $f = 0$. By Proposition 2, we conclude that X is countably thick relative to $\mathcal{K}$. ∎

The next two propositions are similar to [16, 27.2 and 27.3]. We present the proofs for the convenience of the readers.

Proposition 4. *Let $\mathcal{K}$ be a class of simple modules in $\sigma[M]$.*
(1) *Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence in $\sigma[M]$.*

(i) *If N is locally countably thick relative to $\mathcal{K}$, then N' and N'' are locally countably thick relative to $\mathcal{K}$.*

(ii) *If N' is countably thick relative to $\mathcal{K}$ and N'' is locally countably thick relative to $\mathcal{K}$, then N is locally countably thick relative to $\mathcal{K}$.*

(2) *The direct sum of modules in $\sigma[M]$ which are locally countably thick relative to $\mathcal{K}$ is again locally countably thick relative to $\mathcal{K}$.*

Proof. (1)(i) The proof is straightforward.

(1)(ii) Let N' be countably thick relative to $\mathcal{K}$, N'' locally countably thick relative to $\mathcal{K}$, and K a finitely generated submodule of N. Then by using the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & K \cap N' & \to & K & \to & K/K \cap N' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & N' & \to & N & \to & N'' & \to & 0 \end{array}$$

we see that $K \cap N'$ and $K/K \cap N'$ are countably thick relative to $\mathcal{K}$. By Proposition 3, K is also countably thick relative to $\mathcal{K}$.

(2) By Proposition 3, every finite direct sum of countably thick relative to $\mathcal{K}$ modules is countably thick relative to $\mathcal{K}$. If $N, X \in \sigma[M]$ are locally countably thick relative to $\mathcal{K}$, then $N \oplus X$ is locally countably thick relative to $\mathcal{K}$. Let K be a submodule of $N \oplus X$ generated by the elements $(n_1, x_1), \ldots, (n_r, x_r)$ in K (with $n_i \in N, x_i \in X, r \in I\!N$). The submodules $N' = \sum_{i \le r} n_i R \subseteq N$ and $X' = \sum_{i \le r} x_i R \subseteq X$ are countably thick relative to $\mathcal{K}$ by assumption, and hence, $N' \oplus X'$ is countably thick relative to $\mathcal{K}$. Since $K \subseteq N' \oplus X'$, K is also countably thick relative to $\mathcal{K}$.

By induction, we see that every finite direct sum of locally countably thick relative to $\mathcal{K}$ modules in $\sigma[M]$ is locally countably thick relative to $\mathcal{K}$. Then the corresponding assertion is true for arbitrary sums since every finitely generated submodule of it is contained in a finite partial sum. ■

Corollary 5. *For a module M_R and any class $\mathcal{K}$ of simple modules in $\sigma[M]$, the following conditions are equivalent:*

(a) *M is locally countably thick relative to $\mathcal{K}$;*

(b) *every cyclic submodule of M is countably thick relative to $\mathcal{K}$;*

(c) *$M^{(I\!N)}$ is locally countably thick relative to $\mathcal{K}$;*

(d) *$\sigma[M]$ has a set of generators consisting of modules which are countably thick relative to $\mathcal{K}$;*

(e) *every finitely generated (cyclic) module in $\sigma[M]$ is countably thick relative to $\mathcal{K}$;*

(f) *every module in $\sigma[M]$ is locally countably thick relative to $\mathcal{K}$.*

Proof. This follows Proposition 4 and the fact that the finitely generated submodules of $M^{(I\!N)}$ form a set of generators of $\sigma[M]$. ■

3. Thickness vs. Weak Injectivity

Given a module M_R and $Q \in \sigma[M]$, we say that Q is weakly injective in $\sigma[M]$ if, for every finitely generated submodule N of $\widehat{Q}$, there exists a submodule Y of $\widehat{Q}$ such that

$N \subseteq Y \simeq Q$. Note that this notion is different from weakly M-injective as defined in [16, 16.9].

We say that Q is tight in $\sigma[M]$ if every finitely generated submodule N of $\widehat{Q}$ is embeddable in Q, and Q is *weakly tight* in $\sigma[M]$ if every finitely generated submodule N of $\widehat{Q}$ is embeddable in a direct sum of copies of Q.

A module Q_R is said to be weakly injective [12] (tight [11]) if it is weakly injective (tight) in $\sigma[R_R] = \text{Mod} - R$. It is clear that every weakly injective in $\sigma[M]$ is tight in $\sigma[M]$, and every tight module in $\sigma[M]$ is weakly tight in $\sigma[M]$, but weak tightness does not imply tightness. For this, consider the category $\sigma[Q/Z]$, i.e., the torsion Z-modules. Then $X = Q/Z \oplus Z/pZ$ is a cogenerator in $\sigma[Q/Z]$ and $\widehat{X} \simeq Q/Z \oplus Z_{p^\infty}$. Obviously, X is weakly tight in $\sigma[Q/Z]$ (since the category $\sigma[Q/Z]$ is locally artinian). But $Z/p^2Z \oplus Z/p^2Z$ is a finitely generated submodule of $\widehat{X}$ which is not embeddable in X.

By the proof of [15, Lemma 2], we obtain the following:

Lemma 6. *For a module M_R and Q, $L \in \sigma[M]$, we have*
(a) *if Q and L are weakly injective in $\sigma[M]$, then $Q \oplus L$ is also weakly injective in $\sigma[M]$;*
(b) *if Q is an essential submodule of L and Q is weakly injective in $\sigma[M]$, then L is weakly injective in $\sigma[M]$.*

Now, we are in the position to prove the main result.

Theorem 7. *For a module M_R and any class $\mathcal{K}$ of simples in $\sigma[M]$, the following conditions are equivalent:*
(a) *M is locally countably thick relative to $\mathcal{K}$;*
(b) *every direct sum $\bigoplus_\Lambda E_\lambda$ of injectives in $\sigma[M]$, where each E_λ is essential over $Soc_{\mathcal{K}}(E_\lambda)$, is weakly injective in $\sigma[M]$;*
(c) *every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly injectives in $\sigma[M]$, where each M_λ is essential over $Soc_{\mathcal{K}}(M_\lambda)$, is weakly injective in $\sigma[M]$;*
(d) *every direct sum $\bigoplus_\Lambda \widehat{P_\lambda}$, where $P_\lambda \in \mathcal{K}$, is weakly injective in $\sigma[M]$;*
(e) *every direct sum $\bigoplus_\Lambda \widehat{P_\lambda}$, where $P_\lambda \in \mathcal{K}$, is weakly tight in $\sigma[M]$.*

Proof. (a) $\Rightarrow$ (b). Consider $X = \bigoplus_\Lambda E_\lambda$, where E_λ is injective in $\sigma[M]$ for every $\lambda \in \Lambda$ and $\text{Soc}_{\mathcal{K}}(E_\lambda)$ is essential in E_λ.

Let N be a finitely generated submodule of $\widehat{X}$. By the hypothesis, $\text{Soc}_{\mathcal{K}}(N)$ is finitely generated, that is,

$$\text{Soc}_{\mathcal{K}}(N) = P_1 \oplus \cdots \oplus P_n \text{ with } P_i \simeq P_i' \text{ for some } P_i' \in \mathcal{K}\ (1 \leq i \leq n).$$

So

$$\text{Soc}_{\mathcal{K}}(N) \subseteq \text{Soc}_{\mathcal{K}}(\widehat{X}) = \text{Soc}_{\mathcal{K}}(X) \subseteq X,$$

and hence, $\text{Soc}_{\mathcal{K}}(N) \subseteq E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_m}$ for some finite $\{\lambda_1, \ldots, \lambda_m\}$. This implies that $E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_m}$ contains an injective hull E of $\text{Soc}_{\mathcal{K}}(N)$. Since E is injective and contained in X, we may write $X = E \oplus K$ for some submodule K of X. On the other hand, let $\widehat{N}$ be an injective hull of N inside $\widehat{X}$. Then $\widehat{N} = \bigoplus_{i=1}^{n} \widehat{P_{\lambda_i}} \simeq E$. Since $\text{Soc}_{\mathcal{K}}(N)$ is essential in $\widehat{N}$, it follows that $\widehat{N} \cap K = 0$. So let $Y = \widehat{N} \oplus K \simeq E \oplus K = X$, proving that X is weakly injective in $\sigma[M]$.

(b) ⇒ (c). Consider the module $X = \bigoplus_\Lambda M_\lambda$, a direct sum of weakly injectives in $\sigma[M]$, where each M_λ is essential over $\mathrm{Soc}_\mathcal{K}(M)$. Let N be a finitely generated submodule of $\widehat{X}$. By (b), the direct sum $\bigoplus_\Lambda \widehat{M_\lambda}$ is weakly injective in $\sigma[M]$ and $X \subseteq \bigoplus_\Lambda \widehat{M_\lambda} \subseteq \widehat{X}$. Hence, by (b), there exists a submodule $Y \subseteq \widehat{X}$ such that $N \subseteq Y$ and $Y \simeq \bigoplus_\Lambda \widehat{M_\lambda}$. Write $Y = \bigoplus_\Lambda \widehat{Y_\lambda}$ such that $Y_\lambda \simeq M_\lambda$ for all $\lambda \in \Lambda$. Since N is finitely generated, there exists a finite subset $\Gamma \subseteq \Lambda$ such that $N \subseteq \bigoplus_\Gamma \widehat{Y_\lambda} = \widehat{\bigoplus_\Gamma Y_\lambda}$. Since the M'_λ's are weakly injective in $\sigma[M]$, the finite sum $\bigoplus_\Gamma Y_\lambda$ is weakly injective in $\sigma[M]$, and therefore, there exists $X_1 \simeq \bigoplus_\Gamma Y_\lambda \simeq \bigoplus_\Gamma M_\lambda$ such that $N \subseteq X_1 \subseteq \bigoplus_\Gamma Y_\lambda$. But then $N \subseteq X_1 \oplus (\bigoplus_{\Lambda \setminus \Gamma} Y_\lambda) \simeq X$, proving our claim.

(c) ⇒ (d) and (d) ⇒ (e) are trivial.

(e) ⇒ (a). Let C be a finitely generated submodule of M. If $\mathrm{Soc}_\mathcal{K}(C) = 0$, we are done. Suppose $0 \neq \mathrm{Soc}_\mathcal{K}(C) = \bigoplus_\Lambda P_\lambda$, where $P_\lambda \simeq P'_\lambda$ for some $P'_\lambda \in \mathcal{K}$. We show that $\mathrm{Soc}_\mathcal{K}(C)$ is finitely generated.

For this, consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \bigoplus_\Lambda P_\lambda & \xrightarrow{\gamma} & C \\ & & \downarrow \varphi & & \\ & & \bigoplus_\Lambda \widehat{P_\lambda} & & \end{array}$$

where φ and γ are inclusion homomorphisms. By M-injectivity, there exists ψ such that $\psi\gamma = \varphi$. By our hypothesis, $\bigoplus_\Lambda \widehat{P_\lambda}$ is weakly tight in $\sigma[M]$, hence, $\mathrm{Im}\varphi \subseteq \mathrm{Im}\psi$ is finitely $\bigoplus_\Lambda \widehat{P_\lambda}$-cogenerated. Therefore, $\mathrm{Soc}_\mathcal{K}(C)$ is embeddable in $\widehat{P_{\lambda_1}} \oplus \cdots \oplus \widehat{P_{\lambda_n}}$ for some $\lambda_1, \ldots, \lambda_n \in \Lambda$. Since each $\widehat{P_{\lambda_i}}$ is uniform, $\mathrm{Soc}_\mathcal{K}(C)$ has a finite uniform dimension and is therefore finitely generated. ■

For $\mathcal{K} = \{P\}$, P is a simple module in $\sigma[M]$, and we obtain the following:

Corollary 8. *For a module M_R and a simple module P in $\sigma[M]$, the following conditions are equivalent:*

(a) *M is locally countably thick relative to $\{P\}$;*
(b) *every direct sum $\bigoplus_\Lambda E_\lambda$ of injectives in $\sigma[M]$, where each E_λ is essential over $Soc_{\{P\}}(E_\lambda)$, is weakly injective in $\sigma[M]$;*
(c) *every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly injectives in $\sigma[M]$, where each M_λ is essential over $Soc_{\{P\}}(M_\lambda)$, is weakly injective in $\sigma[M]$;*
(d) *$\widehat{P}$ is $\sum$-weakly injective in $\sigma[M]$;*
(e) *$\widehat{P}$ is $\sum$-weakly tight in $\sigma[M]$.*

In the case where Corollary 8 applies for all simple modules, we have the following:

Corollary 9. *For a module M_R, the following conditions are equivalent:*

(a) *M is locally countably distributive;*
(b) *for each simple module P in $\sigma[M]$, every direct sum $\bigoplus_\Lambda E_\lambda$ of injectives in $\sigma[M]$, where each E_λ is essential over $Soc_{\{P\}}(E_\lambda)$, is weakly injective in $\sigma[M]$;*
(c) *for each simple module P in $\sigma[M]$, every direct sum $\bigoplus_\Lambda M_\lambda$ of weakly injectives in $\sigma[M]$, where each M_λ is essential over $Soc_{\{P\}}(M_\lambda)$, is weakly injective in $\sigma[M]$;*
(d) *for each simple module P in $\sigma[M]$, $\widehat{P}$ is $\sum$-weakly injective in $\sigma[M]$;*
(e) *for each simple module P in $\sigma[M]$, $\widehat{P}$ is $\sum$-weakly tight in $\sigma[M]$.*

Finally, taking for $\mathcal{K}$ all simple R-modules (in $\sigma[M]$) and using Theorem 7 and arguments from [2, Theorem], we have the following:

Proposition 10. *For a module M_R, the following conditions are equivalent:*
(a) *M is locally q.f.d.;*
(b) *every direct sum of injectives in $\sigma[M]$ is weakly injective in $\sigma[M]$;*
(c) *direct sums of weakly injectives are weakly injective in $\sigma[M]$;*
(d) *every direct sum $\bigoplus_\Lambda \widehat{P_\lambda}$, where each P_λ is simple in $\sigma[M]$, is weakly injective in $\sigma[M]$;*
(e) *every direct sum $\bigoplus_\Lambda \widehat{P_\lambda}$, where each P_λ is simple in $\sigma[M]$, is weakly tight in $\sigma[M]$.*

Proof. (a) $\Rightarrow$ (b). For this, the proof of (1) $\Rightarrow$ (2) of [2, Theorem] applies.
(b) $\Rightarrow$ (c). This is obtained by the proof of (b) $\Rightarrow$ (c) of Theorem 7.
Other implications follow from Theorem 7. ∎

In the case of $M = R_R$, from Proposition 10, we obtain [2, Theorem].

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