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The Regularity of Spaces of Germs of f-Valued Holomorphic Functions

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Abstract. The present paper is devoted to the study of the regularity of the space $H(K, F)$, where F is a Frechet space with the property (LB_∞) or the property (DN).

The problem of the regularity of the space $H(K)$ of holomorphic germs on a compact set K was investigated by several authors. Chae [3] proved that $H(K)$ is regular for every compact subset K of a Banach space E. When E is a metrizable locally convex space, $H(K)$ is represented as an inductive limit of a sequence of (DF)-spaces. Using a theorem of Grothendieck on bounded subsets in an inductive limit of a sequence of (DF)-spaces, Mujica [7] generalized the result of Chae. Recently, Vogt [16] gave a general characterization for the regularity of the inductive limit of a sequence of Frechet spaces.

The main aim of the present paper is to find some conditions of a given Frechet space F for which the space $H(K, F)$ of germs of F-valued holomorphic functions on compact sets K is regular. These conditions are related to some linear topological invariants. In Secs. 2 and 3 of this paper, we shall prove the following two theorems.

Theorem A. *Let F be a reflexive Frechet space with the property (LB_∞) and E a quotient space of the power series space of infinite type. Then $H(K, F)$ is regular for every unique compact set K in E.*

Theorem B. *A Frechet space F has the property (DN) if and only if $H(K, F)$ is regular for all compact sets K in C^N.*

1. Preliminaries

We shall use standard notations from the theory of locally convex spaces as presented in [9, 10]. All locally convex spaces are assumed to be complex vector spaces and Hausdorff.

1.1. Linear Topological Invariants (LB_∞) and (DN)

Let F be a Frechet space with a fundamental system of seminorms $\{\|\cdot\|_k\}$. We say that F has the properties (DN) and (LB_∞) if the following conditions hold, respectively,
(DN): $\exists p\ \forall q,\ d > 0\ \exists k,\ C > 0\ \forall x \in F$:

$$\|x\|_q^{1+d} \leq C\|x\|_k\|x\|_p^d,$$

(LB_∞): $\forall\{\rho_N > 0\}\ \exists k \in \boldsymbol{N}\ \forall n_0 \in \boldsymbol{N}\ \exists N_0,\ C > 0 : \forall x \in F\ \exists N,\ n_0 < N < N_0$:

$$\|x\|_{n_0}^{1+\rho_N} \leq C\|x\|_N\|x\|_k^{\rho_N}.$$

The properties (LB_∞) and (DN) were introduced and investigated by Vogt [13–16].

1.2. Sequence Spaces

Let $A = (a_{jk})$ be a Köthe matrix satisfying the conditions given in [9, 6.1]. We denote by $\Lambda(A)$ the Frechet space

$$\Lambda(A) = \{x = (x_j) \in C^N : p_k(x) = \sum_{j\geq 1} |x_j|\, a_{jk} < \infty,\quad \forall k \geq 1\}.$$

For $0 < R \leq +\infty$, we write $\Lambda_R(\alpha)$ instead of $\Lambda(A)$ if $a_{jk} = r_k^{\alpha_j}$, where $\alpha = (\alpha_j)$ is an increasing sequence of positive real numbers with $\lim_j \alpha_j = +\infty$ and $\{r_k\}$ is an increasing sequence convergent to R. $\Lambda_R(\alpha)$ is called a power series space of finite type if $R < \infty$, and of infinite type if $R = \infty$.

1.3. Holomorphic Functions

Let E and F be locally convex spaces and D an open subset of E. A function $f : D \to F$ is called holomorphic if f is continuous and Gateaux holomorphic. By $H(D, F)$, we denote the vector space of holomorphic functions on D with values in F. For details concerning holomorphic functions, we refer the reader to [8, 10].

A seminorm ρ on $H(D, F)$ is said to be τ_ω,-continuous if there exists a compact set K in D and a continuous seminorm α on F such that, for every neighborhood V of K in D, there exists $C(V) > 0$ such that

$$\rho(f) \leq C(V) \sup_{z\in V} \alpha(f(z)) \qquad \forall f \in H(D, F).$$

Given K a compact set in E. By $H(K, F)$, we denote the space of germs of F-valued holomorphic functions on K equipped with the inductive topology

$$H(K, F) = \operatorname*{limind}_{U\supset K}(H(U, F); \tau_\omega).$$

It is known [7] that

$$H(K) \cong \operatorname*{limind}_{U\supset K} H^\infty(U),$$

where $H^\infty(U)$ denotes the Banach space of bounded holomorphic functions on U.

2. Proof of Theorem A

To prove Theorem A, we first establish the following:

Lemma 2.1. *Let F be a Frechet space with the property (LB_∞) and B a Banach space. Then the space $L(B, F)$ of all continuous linear maps from B into F has the property (LB_∞).*

Proof. Given a sequence of positive numbers $\{\rho_N\}$, choose $k \in N$ such that property (LB_∞) is satisfied. Then

$$\begin{aligned} \|f\|_{n_0}^{1+\rho_N} &= sup\{\|f(x)\|_{n_0}^{1+\rho_N} : \|x\| \le 1\} \\ &\le C \max_{n_0 \le N \le N_0} \sup\{\|f(x)\|_N . \|f(x)\|_k^{\rho_N} : \|x\| \le 1\} \\ &\le C \max_{n_0 \le N \le N_0} \|f\|_N . \|f\|_k^{\rho_N} \quad \text{for } f \in L(B, F). \end{aligned}$$

Hence, $L(B, F)$ has property (LB_∞). ■

Lemma 2.2. *Let F be a Frechet space with the property (LB_∞) and E a quotient space of $\Lambda_\infty(\alpha)$. Then*

$$L([H(K, B)]^*, F) = LB([H(K, B)]^*, F)$$

for every Banach space B and every compact set K in E.

Proof. Given a continuous linear map

$$\eta : [H(K, B)]^* \to F.$$

Because $[H(K, B)]^* \cong [H(K)]^* \hat{\otimes}_\pi B^*$, η induces a continuous linear map

$$\hat{\eta} : [H(K)]^* \to L(B^*, F).$$

From Meise and Vogt [6], it follows that $[H(K)]^*$ is a quotient space of $\Lambda_\infty(\beta(\alpha))$. By Lemma 2.1 and [14], it implies that $\hat{\eta}$ is bounded on a zero neighborhood U in $[H(K)]^*$. This yields that η is bounded on $\overline{\mathrm{conv}}\,(U \otimes V)$, a zero neighborhood in $[H(K)]^* \hat{\otimes}_\pi B^*$, where V is the unit ball in B^*. This completes the proof. ■

We mention the following without proof.

Lemma 2.3. *Let f be a bounded function from an open set D in a locally convex space E into the Banach space $l^\infty(I)$ and let coordinate functions f_α be holomorphic. Then f is holomorphic.*

Now, we can prove Theorem A.

Proof of Theorem A. Given a bounded family $\{f_\alpha\}_{\alpha \in I}$ in $H(K, F)$, consider the linear map

$$s : F^*_{bor} \to H(K, l^\infty(I))$$

defined by

$$s(u) = (u \circ f_\alpha)_{\alpha \in I}.$$

From the uniqueness of K and Lemma 2.3, it implies that $s(u)$ is correctly defined.

(i) We first check that s is bounded. Indeed, let B be a bounded set in F^*. Take $k \geq 1$ such that B is contained and bounded in F_k^*, where F_k is the Banach space associated to $\|.\|_k$. Let $\omega_k : F \to F_k$ be the canonical map. By the regularity of $H(K, F_k)$[12] and the boundedness of $\{\omega_k f_\alpha\}_{\alpha \in I}$, we can find a neighborhood V of K in E such that $\{\omega_k f_\alpha\}_{\alpha \in I}$ is contained and bounded in $H^\infty(V, F_k)$, the Banach space of F_k-valued bounded holomorphic functions on V. Hence, $s(B)$ is contained and bounded in $H^\infty(V, l^\infty(I))$. Therefore, s is continuous.

(ii) By Lemma 2.2, the map $s^* : [H(K, l^\infty(I))]^* \to [F_{bor}^*]^* \cong F$ is of type (LB). It follows that s^{**}, and hence, s is also of type (LB). Thus, we can find a neighborhood W of zero in F_{bor}^* for which there exists, for every $u \in F^*$, a function $\hat{s}(u)$ in $H^\infty(V, l^\infty(I))$ such that

(1) $\hat{s}(u) = s(u)$ on a neighborhood of K in V,

(2) $\{\hat{s}(u)\}_{u \in W}$ is bounded in $H^\infty(V, l^\infty(I))$.

Now, for each $\alpha \in I$, we define a holomorphic function

$$g_\alpha : V \to [F_{bor}^*]^* \cong F$$

by

$$g_\alpha(z)(u) = u \circ f_\alpha(z) \quad \text{for } z \in V \text{ and } u \in F_{bor}^*.$$

By (2), $\{g_\alpha\}_{\alpha \in I}$ is bounded in $H^\infty(V, F)$. Thus, $\{f_\alpha\}_{\alpha \in I}$ is contained and bounded in $(H^\infty(V, F); \tau_\omega)$.

Theorem A is proved. ∎

3. Proof of Theorem B

Lemma 3.1. *Let F be a Frechet space having a continuous norm and let $H(0, F)$, where $0 \in C^N$, be regular. Then $H(K, F)$ is regular for all compact sets $K \subset C^N$.*

Proof. Let $\{f_\alpha\}_{\alpha \in A}$ be a bounded family in $H(K, F)$. By this hypothesis, we can find a finite open-convex covering $\{U_{z_j}, \ j = 1, ..., m, \ z_j \in K\}$ of K such that, for each $j = 1, ..., m$, there exists a bounded set $\{g_{\alpha,j}\}_{\alpha \in A}$ in $H(U_{z_j}, F)$ satisfying

$$g_{\alpha,j}^{z_j} = f_\alpha^{z_j} \quad \text{for all } \alpha \in A,$$

where $g_{\alpha,j}^{z_j}$ and $f_\alpha^{z_j}$ are the germs of $g_{\alpha,j}$ and f_α at z_j, respectively.

On the other hand, by the regularity of $H(K, F_p)$[12], where $\|.\|_p$ is a continuous norm on F, there exists a neighborhood W of K such that

$$\omega_p f_\alpha \in H(W, F_p) \quad \text{for all } \alpha \in A.$$

Since

$$\omega_p g_{\alpha,j}|_{U_{z_j} \cap W} = \omega_p f_\alpha|_{U_{z_j} \cap W},$$

we have

$$\omega_p g_{\alpha,j}|_{U_{z_j} \cap U_{z_i}} = \omega_p g_{\alpha,j}|_{U_{z_j} \cap U_{z_i}} \quad \forall j, \ i = 1, ..., m.$$

Hence,

$$g_{\alpha,i}|_{U_{z_j} \cap U_{z_i}} = g_{\alpha,j}|_{U_{z_j} \cap U_{z_i}} \quad \forall j, \ i = 1, ..., m.$$

This yields that, for each $\alpha \in A$, $\{g_{\alpha,j}\}_{j=1}^m$ defines a holomorphic function $g_\alpha \in H(U, F)$ inducing f_α, where $U = \bigcup_{j=1}^m U_{z_j}$.

Obviously, $\{g_\alpha\}_{\alpha\in A}$, and hence, $\{f_\alpha\}_{\alpha\in A}$ is contained and bounded in $H(U, F)$. The lemma is proved. ■

Proof of Theorem B. Assume F is a Frechet space with $F \in (DN)$. Then F can be considered as a subspace of the space

$$B\hat{\otimes}_\pi s \cong \{(y_p) \subset B : \sum_{p\geq 1} \|y_p\| p^n < \infty, \forall n \geq 1\}$$

for some Banach space B, where s denotes the space of rapidly decreasing sequences. By Lemma 3.1, it suffices to show that $H(0, B\hat{\otimes}_\pi s)$ is regular, where $0 \in C^N$. We may assume $N = 1$.

Writing each $\sigma \in H^\infty(r\Delta, B\hat{\otimes}_\pi s)$ in the form

$$\sigma(z) = \left\{\sum_{j\geq 1} \xi_{jp} z^j\right\}_{p\geq 1},$$

we have

$$H(0, B\hat{\otimes}_\pi s) \cong \lim_k \operatorname{ind} E_k,$$

where

$$E_k = \left\{\sigma = (\xi_{jp}) \subset B : \|\sigma\|_{k,n} := \sum_{j,p} \|\xi_{jp}\| \left(\frac{1}{k}\right)^j p^n < \infty, \quad \forall n \geq 1\right\}.$$

In view of the regular characterization of Vogt [16], we have to check the following:

$$\forall \mu \ \exists k, n \ \forall K, m \ \exists N, S \ \forall \sigma \in H^\infty\left(\frac{1}{\mu}\Delta, B\hat{\otimes}_\pi s\right) \qquad (1)$$
$$\|\sigma\|_{k,m} \leq S(\|\sigma\|_{\mu,n} + \|\sigma\|_{K,N}).$$

Given $\mu \geq 1$, choose $k = 2\mu$ and $n = 1$. We first note that

$$\forall K, m \ \exists N, S \ \forall j, p : \left(\frac{1}{2\mu}\right)^j p^m \leq S\left(\left(\frac{1}{\mu}\right)^j p + \left(\frac{1}{K}\right)^j p^N\right). \qquad (2)$$

Indeed, (2) obviously holds for (j, p) satisfying $p^m \leq 2^j$.

In the case $p^m > 2^j$, it is easy to see that (2) holds if

$$N > \frac{m\log K}{\log 2} + m.$$

From (2), we have

$$\|\sigma\|_{2\mu,m} = \sum_{j,p} \|\xi_{jp}\| \left(\frac{1}{2\mu}\right)^j p^m$$
$$\leq S \sum_{j,p} \left\{\|\xi_{jp}\| \left(\left(\frac{1}{\mu}\right)^j p + \left(\frac{1}{K}\right)^j p^N\right)\right\}$$
$$\leq S(\|\sigma\|_{\mu,1} + \|\sigma\|_{K,N}).$$

Conversely, assume $H(K,F)$ is regular for all compact sets in C^N. In particular, $H(\bar{\Delta}, F)$ is regular. By [16], we have

$$\forall\mu\ \exists k,n\ \forall m,K\ \exists N,S:$$

$$\|\sigma\|_{k,m} \le S(\|\sigma\|_{\mu,n} + \|\sigma\|_{K,N}) \quad \forall\sigma \in H^\infty\left(\frac{k+1}{k}\Delta, F\right), \tag{3}$$

where

$$\|\sigma\|_{k,m} = \sup\{\|\sigma(z)\|_m : z \in \frac{k+1}{k}\Delta\}.$$

By applying (3) to $z^j x$, for $x \in F$, we have

$$\left(\frac{k+1}{k}\right)^j \|x\|_m \le S\left(\left(\frac{\mu+1}{\mu}\right)^j \|x\|_n + \left(\frac{K+1}{K}\right)^j \|x\|_N\right).$$

This inequality yields that, for $\mu = 1$, $\exists k, n$:

$$\|x\|_m \le S(e^{(\sigma_k-\sigma_1)j}\|x\|_n + e^{(\sigma_k-\sigma_{k+1})j}\|x\|_N),$$

where

$$\sigma_k = \log\frac{k}{k+1} \quad \text{for } k \ge 1.$$

Given $r \ge e^{(\sigma_k-\sigma_1)}$, we choose j such that

$$(\sigma_k-\sigma_1)j \le \log r \le (\sigma_k-\sigma_1)(j+1) \le 2(\sigma_k-\sigma_1)j,$$

which gives

$$\|x\|_m \le S(r\|x\|_n + \frac{1}{r^\delta}\|x\|_N)$$

with

$$\delta = \frac{1}{2}\frac{(\sigma_{k+1}-\sigma_k)}{(\sigma_k-\sigma_1)}.$$

We can increase S so that the inequality holds for all $r > 0$. A calculation of the minimum of the function of r on the right side gives

$$\|x\|_m \le C^{\frac{1}{1+\delta}}\|x\|_N^{\frac{1}{1+\delta}}\|x\|_n^{\frac{\delta}{1+\delta}}.$$

This means that $F \in (DN)$. Theorem B is proved. ■

To complete the paper, we prove the following:

Proposition 3.2. *Let F be a Frechet space such that $H(K,F)$ is regular for some balanced compact set K in a Frechet space E. Then F has the continuous norm.*

Proof. By [1], it suffices to show that $H(K,\omega)$, where ω is the space of all complex number sequences, is not regular for all balanced compact sets K in a Frechet space E. Choose a balanced neighborhood basis $\{U_k\}$ of K and a sequence $\delta_k > 1$ such that

$$\delta_k U_{k+1} \subset U_k \quad \text{for all } k \geq 1.$$

For $k = 1$, choose $\sigma_1 \in H(U_1) \setminus H^\infty(U_1)$. Since K is compact, there exists $k_1 > 1$ for which $\sigma_1 \in H^\infty(U_{k_1})$. Without loss of generality, we may assume $k_1 = 2$. By continuing this process, we obtain a sequence of holomorphic functions $\sigma_k \in H(U_k)$ such that

$$\sigma_k \notin H^\infty(U_k) \quad \text{and} \quad \sigma_k \in H^\infty(U_{k+1}) \quad \text{for } k \geq 1.$$

Consider the sequence $\{\hat{\sigma}_k\} \subset H(K,\omega)$ given by

$$\hat{\sigma}_k = (0, \dots, 0, \sigma_k, 0, \dots).$$

Obviously,

$$\{\hat{\sigma}_k\} \not\subset H(U_j,\omega) \quad \text{for } j \geq 1.$$

It remains to be checked that $\{\hat{\sigma}_k\}$ is bounded in $H(K,\omega)$. Indeed, given

$$W = W_1 + \dots + W_k + \dots$$

a neighborhood of zero in $H(K,\omega)$, where

$$W_k = \{\varphi \in H(U_k,\omega) : \rho_k(\varphi) \leq 1\}$$

and ρ_k is a τ_ω-continuous seminorm on $H(U_k,\omega)$, by the definition of a τ_ω-continuous seminorm, for each $k \geq 1$, we can find $n_k \geq 1$ and $\epsilon_k > 0$ such that

$$\{\varphi = (\varphi_j) \in H(U_k,\omega) : \max_{1\leq j\leq n_k} \|\varphi_j\| < \epsilon_k\} \subset W_k .$$

We may assume $n_k \leq n_{k+1}$ for $k \geq 1$. Since $\delta_{k+1} U_{k+2} \subset U_{k+1}$, $\delta_{k+1} > 1$, we can write for each $k \geq 1$

$$\sigma_k = P_k + Q_k ,$$

where P_k is a polynomial on E and $Q_k \in H(U_k)$ with

$$\|Q_k\|_{U_{k+2}} < \epsilon_{k+2} .$$

From the relations

$$(0, \dots, 0, P_k, 0, \dots) \in W_1 \quad \text{for } k > n_1$$

and

$$(0, \dots, 0, Q_k, 0, \dots) \in W_{k+2} \quad \text{for } k \geq 1,$$

we have $\hat{\sigma}_k \in W$ for $k > n_1$. Hence $\hat{\sigma}_k \to 0$ in $H(K,\omega)$ and this completes the proof. ■

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