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On the Composition of Random Operators on Banach Spaces*

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Abstract. In this paper, the composition of random operators between Banach spaces is defined in a natural manner. Unlike deterministic operators, the composition of random operators need not exist. Some conditions for the existence of the composition are provided.

1. Introduction

Let X and Y be separable Banach spaces. By a random operator A from X into Y, we mean a linear continuous mapping from X into the space $L_0^Y(\Omega)$ of Y-valued random variables with the topology of the convergence in probability. For the motion of the study of random operators, see [16]. Some aspects of the theory of random operators acting between Banach spaces were investigated in [1, 10, 13–15].

This paper which is a continuation of [16] is devoted to the notion of the composition of two random operators acting between Banach spaces. It seems natural to consider the composition AB of two random operators A and B as a transformation obtained by performing B then performing A. But under the original definition, the random operator A cannot act on random variables while the range of B consists of random variables with values in the domain of A. Hence, the first discussion must be to give a reasonable definition to the action of A to some random variables taking values on the domain of A. In other words, given a random operator A from X into Y, the problem is to extend the domain of A to some class of X-valued random variables. Of course different procedures may be proposed but the aim will be that the extension must be as wide as possible and at the same time it should enjoy many good properties similar to that of A.

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In Sec. 2, a procedure of the extension is introduced in a natural way in the case $X = l_s$ $(1 \le s < \infty)$. This enables us to give in Sec. 3 the definition of the composition of two random operators A and B, where A is the random operator from X into Y, B the random operator from Z into X, and Y and Z are arbitrary separable Banach spaces. We determine some conditions for the existence of the composition and give some examples.

2. Action of Random Operators on l_s-valued Random Variables

Throughout this paper, $X = l_s$ $(1 \le s < \infty)$, $e = (e_n)$ is the standard basis in X. Recall that by a random operator A from X into Y, we mean a linear continuous mapping from X into $L_0^Y(\Omega)$ where $L_0^Y(\Omega)$ stands for the set of all Y-valued random variables equipped with the topology of convergence in probability. The set of all random operators from X into Y is denoted by $L(\Omega, X, Y)$. Since $A : X \to L_0^Y(\Omega)$ is linear and continuous, we have

$$Ax = \sum_{n=1}^{\infty} (x, e_n) Ae_n \quad \text{in } L_0^Y(\Omega) .$$

This suggests the following definition.

Definition 2.1. *An X-valued variable u is said to be A-applicable if the series*

$$\sum_{n=1}^{\infty} (u, e_n) Ae_n$$

converges in $L_0^Y(\Omega)$. In this case, the sum of the series is denoted by Au. The set of all A-applicable random variables is denoted by $\mathcal{D}(A)$.

Using the same argument as given in the proofs of [16, Propositions 4.2–4.4], we get the following propositions.

Proposition 2.2. *Assume a random operator $A \in L(\Omega, X, Y)$ has a modification whose sample paths belong to $L(X, Y)$. Then every X-valued random variable u is A-applicable.*

Proposition 2.3. *If $u_1, u_2, \ldots, u_n$ belong to $\mathcal{D}(A)$ and $\xi_1, \xi_2, \ldots, \xi_n$ are real-valued random variables, then the linear combination of the form $u = \sum_{i=1}^{n} \xi_i u_i$ also belongs to $\mathcal{D}(A)$ and we have*

$$Au = \sum_{i=1}^{n} \xi_i Au_i .$$

Proposition 2.4. *If $u \in L_0^X(\Omega)$ is a countably valued random variable, then u is A-applicable and Au can be computed by the direct substitution.*

Theorem 2.5. *Let $u \in L_0^X(\Omega)$ and the sequence $(Ae_n)_{n=1}^{\infty}$ be independent. Then u is A-applicable.*

Proof. By the independence of u and the sequence (Ae_n), we have

$$P\left\{\left\|\sum_{i=m}^{n}(u,e_i)Ae_i\right\|>\varepsilon\right\}=\int_X P\left\{\left\|\sum_{i=m}^{n}(u,e_i)Ae_i\right\|>\varepsilon\right\}dP_u(x),$$

where P_u is the distribution of u on X. Because for each $x\in X$

$$\lim_{m,n\to\infty} P\left\{\left\|\sum_{i=m}^{n}(x,e_i)Ae_i\right\|>\varepsilon\right\}=0, \tag{2.1}$$

by passing to the limit under the integral (2.1), we get

$$\lim_{m,n\to\infty} P\left\{\left\|\sum_{i=m}^{n}(u,e_i)Ae_i\right\|>\varepsilon\right\}=0,$$

i.e., the series $\sum_{i=1}^{\infty}(u,e_i)Ae_i$ converges in $L_0^Y(\Omega)$. ■

From now on, for brevity, we denote the random variable (u,e_n) by u_n and (x,e_n) by x_n. We associate to A a family of increasing σ-algebra $(\mathcal{F}_n)$ as follows: $\mathcal{F}_n$ is the σ-algebra generated by $Ae_1,\ldots,Ae_n$. An X-valued random variable u is said to be predictable (with respect to A) if, for each $n>1$, the nth coordinate u_n is $\mathcal{F}_{n-1}$-measurable.

Theorem 2.6. *Let H be a Hilbert space and A a random operator from $X=l_s$ into H. Suppose the r.v.'s (Ae_n) are independent. Then each X-valued predictable random variable u is A-applicable.*

The proof is based on the following lemma, which is an extension of [3, Theorem 2] to the case of Hilbert space-valued r.v.'s.

Lemma 2.7. *Let (ξ_n) be a sequence of H-valued r.v.'s and $\mathcal{F}_n$ the σ-algebra generated by $\xi_1,\ldots,\xi_n$. Suppose $\mu_n(\omega)$ is the regular conditional distribution of ξ_n given $\mathcal{F}_{n-1}$. Then the series $\sum_{n=1}^{\infty}\xi_n$ converges a.s. if, for almost ω, the sequence $\{\mu_n(\omega)\}$ sums in probability in the following sense: For each sequence (ζ_n) of H-valued independent r.v.'s defined on another probability space $(\omega',\mathcal{F}',P')$, such that the distribution of $\zeta_n(\omega')$ is $\mu_n(\omega)$ $(n=1,2,\ldots)$, the series $\sum_{n=1}^{\infty}\zeta_n$ converges in $L_0^H(\Omega)$.*

Lemma 2.7 can be proved by the same argument as given in the proof of [3, Theorem 2] by using the Kolmogorov three-series theorem for independent r.v.'s taking values in Hilbert spaces (see [12]).

Proof of Theorem 2.6. Let $\mu(\omega)$ be the regular conditional distribution of u_nAe_n given $\mathcal{F}_{n-1}$. Since u_n is $\mathcal{F}_{n-1}$-measurable and Ae_n is independent of $\mathcal{F}_{n-1}$, we have

$$\mu_n(\omega)(E)=P\{u_nAe_n\in E/\mathcal{F}_{n-1}\}=P\{\omega':u_n(\omega)Ae_n(\omega')\in E\}.$$

Let $\nu_n(x)$ be the distribution of $x_n A e_n$. We have

$$\nu_n(x)(E) = P\{\omega' : x_n A e_n(\omega') \in E\}.$$

Consequently,

$$\mu_n(\omega) = \nu_n(u(\omega)). \tag{2.2}$$

Because, for each $x \in X$, the r.v.'s $(x_n A e_n)$ are independent and the series $\sum_{n=1}^{\infty} x_n A e_n$ converges in probability, from (2.2), it follows that the sequence $\{\mu_n(\omega)\}$ sums in probability. By Lemma 2.7, we conclude that the series $\sum_{n=1}^{\infty} u_n A e_n$ converges a.s., that is, u is A-applicable. ∎

In the case where Y is a Banach space, we have to impose some conditions about the smoothness of Y as well as some additional conditions on A.

A Banach space Y is said to be p-uniformly smooth ($1 \leq p \leq 2$) if the modulus of smoothness $\rho(t)$ satisfies $\rho(t) = O(t^p)$, where the modulus of smoothness is defined by

$$\rho(t) = \sup_{\substack{\|x\|=1 \\ \|y\|=t}} \left\{ \frac{\|x+y\| + \|x-y\| - 2}{2} \right\}.$$

A Banach space Y is said to be p-smoothable if Y is isomorphic to a p-uniformly smooth space. Szulga [11] characterized p-smoothable Banach space by the fact that the conditional three-series theorem for the sequence of Y-valued r.v.'s holds if and only if Y is sufficiently smooth, namely, we have the following result.

Theorem 2.8. (see [11, Theorem 2.1]) *The following conditions are equivalent:*
(1) *A Banach space Y is p-smoothable.*
(2) *For any Y-valued sequence (ζ_n), the a.s. convergence of the following three series*

$$\sum_{n=1}^{\infty} P\{\|\zeta_n\| > c/\mathcal{F}_{n-1}\}$$

$$\sum_{n=1}^{\infty} E\{\zeta_n^c/\mathcal{F}_{n-1}\}$$

$$\sum_{n=1}^{\infty} E\{\|\zeta_n^c - E(\zeta_n^c/\mathcal{F}_{n-1})\|^p/\mathcal{F}_{n-1}\}$$

implies the a.s. convergence of the series $\sum_{n=1}^{\infty} \zeta_n$, where $\zeta_n^c = \zeta_n 1_{\{\|\zeta_n\| \leq c\}}$ and $\mathcal{F}_n$ denotes the σ-algebra generated by $\zeta_1, \ldots, \zeta_n$.

A random operator A from X into Y is called a symmetric p-stable random operator ($0 < p \leq 2$) if, for every finite sequence $\{(x_k, y_k)\}_{k=1}^n$ in $X \times Y'$, the joint distribution of $\{(Ax_k, y_k)\}_{k=1}^n$ is symmetric p-stable.

Theorem 2.9. *Let A be a symmetric p-stable random operator from $X = l_s$ into Y and Y a q-smoothable Banach space, where $q = 2$ if $p = 2$ and $q > p$ if $0 < p < 2$. Suppose the r.v.'s Ae_n are independent. Then each X-valued predictable random variable u satisfying*

$$\sum_{n=1}^{\infty} |u_n|^p < \infty \quad a.s.$$

is A-applicable.

Proof. For brevity, we write Z_n for Ae_n. At first, we shall show that

$$\sum_{n=1}^{\infty} P\{\|u_n Z_n\| > c/\mathcal{F}_{n-1}\} < \infty \quad \text{a.s.} \tag{2.3}$$

$$\sum_{n=1}^{\infty} E\{u_n Z_n 1_{\{\|u_n Z_n\| \le c\}}/\mathcal{F}_{n-1}\} = 0 \quad \text{a.s.} \tag{2.4}$$

Indeed, because $\sum_{n=1}^{\infty} u_n Z_n$ converges a.s., by the Borel-Cantelli, we have

$$\sum_{n=1}^{\infty} P\{\|u_n Z_n\| > c\} < \infty \quad \text{for each } x \in X.$$

By the assumption that u_n is $\mathcal{F}_{n-1}$-measurable, Z_n is symmetric and independent of $\mathcal{F}_{n-1}$, we get

$$\sum_{n=1}^{\infty} P\{\|u_n Z_n\| > c/\mathcal{F}_{n-1}\} = \sum_{n=1}^{\infty} P\{\omega' : \|u_n(\omega) Z_n(\omega')\| > c\} < \infty \quad \text{a.s.}$$

$$E\{u_n Z_n 1_{\{\|u_n Z_n\| \le c\}}/\mathcal{F}_{n-1}\} = u_n(\omega) \int_{\{\omega' : \|Z_n(\omega')\| \le c/u_n(\omega)\}} Z_n(\omega')\, dP(\omega') = 0,$$

which proves (2.3) and (2.4).

Now, we shall show that

$$\sum_{n=1}^{\infty} E\{\|u_n Z_n 1_{\{\|u_n Z_n\| \le c\}}\|^q/\mathcal{F}_{n-1}\} < \infty \quad \text{a.s.} \tag{2.5}$$

(a) The case $p = 2$ (the Gaussian case). For each $x \in X$, Ax is a Y-valued Gaussian random variable so $Ax \in L_2^Y(\Omega)$. Hence, A may be considered as a linear mapping from X into $L_2^Y(\Omega)$. By the closed graph theorem, A is continuous, which implies that $\sup_n E\|Ae_n\|^2 = K < \infty$. Now, by assumption, $q = 2$ and we have

$$\sum_{n=1}^{\infty} E\{\|u_n Z_n 1_{\{\|u_n Z_n\| \le c\}}\|^2/\mathcal{F}_{n-1}\} = \sum_{n=1}^{\infty} |u_n|^2 E\{\|Z_n\|^2 1_{\{\|u_n Z_n\| \le c\}}/\mathcal{F}_{n-1}\}$$

$$\le \sum_{n=1}^{\infty} |u_n|^2 E\{\|Z_n\|^2/\mathcal{F}_{n-1}\} = \sum_{n=1}^{\infty} |u_n|^2 E\|Z_n\|^2 \le K \sum_{n=1}^{\infty} |u_n|^2 < \infty \quad \text{a.s.}$$

(b) The case $0 < p < 2$. We need the following lemma:

Lemma 2.10. *There exists a constant $K > 0$ such that*

$$\sup_{n\geq 1} P\{\|Z_n\| > c\} \leq K c^{-p}. \tag{2.6}$$

Proof of Lemma. For each $x \in X$, Ax is a Y-valued, p-stable random variable so $Ax \in L_r^Y(\Omega)$ $(r < p)$. By the closed graph theorem, A can be viewed as a linear continuous mapping from X into $L_r^Y(\Omega)$, which implies that

$$\sup_n E\|Ae_n\|^r = K_1 < \infty. \tag{2.7}$$

Since Y is q-smoothable, it is of type q. By [18, Proposition V.5.1], this implies that Y is of stable type p. Hence, in view of [5, Corollary 7.3.5, Proposition 7.5.4], there exists a constant $K_2 > 0$ depending only on r and p such that

$$P\{\|Ae_n\| > c\} \leq K_2 c^{-p} E\{\|Ae_n\|^r\}^{p/r}. \tag{2.8}$$

Combining (2.7) and (2.8), we get (2.6) as claimed.

Now, we prove (2.5). Using the independence of Z_n and $\mathcal{F}_{n-1}$ together with the $\mathcal{F}_{n-1}$-measurability of u_n, we have

$$\begin{aligned} E\{\|u_n Z_n 1_{\{\|u_n Z_n\|\leq c\}}\|^q / \mathcal{F}_{n-1}\} &= |u_n|^q E\{\|Z_n\|^q 1_{\{\|u_n Z_n\|\leq c\}} / \mathcal{F}_{n-1}\} \\ &= |u_n|^q \int_0^{c/|u_n(\omega)|} t^q \, dP\{\|Z_n\| < t\}. \end{aligned}$$

Integration by parts and the estimation (2.6) yield

$$\begin{aligned} \int_0^{c/|u_n(\omega)|} t^q \, dP\{\|Z_n\| < t\} &\leq q \int_0^{c/|u_n(\omega)|} t^{q-1} P\{\|Z_n\| > t\}\, dt \\ &\leq Kq \int_0^{c/|u_n(\omega)|} t^{q-p-1}\, dt = \frac{Kq}{q-p} (c/|u_n(\omega)|)^{q-p}. \end{aligned}$$

Consequently,

$$\begin{aligned} \sum_{n=1}^{\infty} E\{\|u_n Z_n 1_{\{\|u_n Z_n\|\leq c\}}\|^q / \mathcal{F}_{n-1}\} &\leq \frac{Kq}{q-p} \sum_{n=1}^{\infty} |u_n(\omega)|^q (c/|u_n(\omega)|)^{q-p} \\ &= \frac{Kqc^{q-p}}{q-p} \sum_{n=1}^{\infty} |u_n(\omega)|^p < \infty \quad \text{a.s.} \end{aligned} \tag{2.9}$$

which shows the a.s. convergence of the series (2.5).

Having established the a.s. convergence of three series (2.3)–(2.5), we can conclude that the series $\sum_{n=1}^{\infty} u_n A_n$ converges a.s. by Theorem 2.8. Theorem 2.9 is proved. ■

Corollary 2.11. *Let A be a symmetric p-stable random operator from $X = l_p$ $(1 \le p \le 2)$ into a q-smoothable Banach space Y, where $q = 2$ if $p = 2$ and $q > p$ if $p < 2$. Suppose the r.v.'s (Ae_n) are independent. Then each X-valued predictable random variable u is A-applicable.*

3. The Composition of Random Operators

Definition 3.1. *Let Y and Z be separable Banach spaces, and let $A \in L(\Omega, X, Y)$, $B \in L(\Omega, Z, X)$. We say that the composition AB exists if, for each $x \in Z$, Bx is A-applicable. In this case, the composition is defined by*

$$(AB)z = A(Bz).$$

Theorem 3.2. *The composition AB, if it exists, is a random operator from Z into Y.*

Proof. By definition we have

$$(AB)z = \sum_{n=1}^{\infty} (Bz, e_n) Ae_n \, .$$

Define the mapping $C_n : Z \to L_0^Y(\Omega)$ by

$$C_n z = \sum_{i=1}^{n} (Bz, e_i) Ae_i \, .$$

It is easy to prove that C_n is linear and continuous. Since $C_n z = (AB)z$ in $L_0^Y(\Omega)$ for every $z \in Z$, it follows from the Banach-Steinhaus theorem that AB is also a linear continuous mapping, i.e., it is a random operator.

Proposition 2.2 and Theorem 2.5 allow us to obtain the following conditions for which the composition exists.

Theorem 3.3.
(a) *Assume the random operator A has a modification whose sample paths are in $L(X, Y)$. Then the composition AB exists for every $B \in L(\Omega, Z, X)$.*
(b) *If A and B are independent random operators, then the composition always exists.*

A random operator B is said to be A-predictable if, for each $z \in Z$, the X-valued random variable Bz is predictable with respect to A. As a consequence of Theorems 2.6 and 2.7, we obtain

Theorem 3.4.
(a) *Let Y be a Hilbert space and assume the random variables (Ae_n) are independent. Then AB exists for each A-predictable random operator $B \in L(\Omega, Z, X)$.*
(b) *Let Y be a q-smoothable Banach space and assume A is a symmetric p-stable random operator, where $q = 2$ if $p = 2$, $p < q < 2$ if $0 < p < 2$ and the random variables (Ae_n) are independent. Then AB exists for each A-predictable random operator $B \in (\Omega, Z, X)$ satisfying:*

For each $z \in Z$, $\sum_{n=1}^{\infty} |(Bz, e_n)|^p < \infty$ *a.s.*

We close this paper by exhibiting two symmetric p-stable random operators A, B from l_s into l_s $(1 \le s < p)$ for which AA, BA exist but BB and AB do not.

First, we need the following lemma.

Lemma 3.5. [18] *Let* (c_n) *be a sequence of real-numbers and* (α_n) *the* p*-stable standard sequence, i.e., the sequence of i.i.d. real-valued random variables such that* $E\exp(it\alpha_1) = \exp(-|t|^p)$. *Then the series* $\sum_{n=1}^{\infty} c_n\alpha_n$ *converges a.s. if and only if* $(c_n) \in l_p$ *and the series* $\sum_{n=1}^{\infty} |c_n\alpha_n|^s$ *converges a.s. if and only if*

$$(c_n) \in l_s \quad \text{in the case} \quad s < p < 2,$$
$$(c_n) \in l_p \quad \text{in the case} \quad p < s \le \infty.$$

Now, let $1 \le s < p < 2$ and $a = (a_i) \in l_s$. Consider random operators A and B from l_s into l_s defined by

$$Ax = a\sum_{i=1}^{\infty} \alpha_i x_i \quad \text{if } x = (x_i) \in l_s\,,$$

$$Bx = \sum_{i=1}^{\infty} \alpha_i x_i e_i \quad \text{if } x = (x_i) \in l_s\,.$$

By Lemma 3.5, the series $\sum_{i=1}^{\infty} \alpha_i x_i$ and $\sum_{i=1}^{\infty} |\alpha_i x_i|^s$ converge a.s. so A and B are well defined.

(i) Let $u = Ax$. Then $\sum_{i=1}^{\infty}(u, e_i)Ae_i = \left(\sum_{i=1}^{\infty} \alpha_i x_i\right)a\sum_{i=1}^{\infty} a_i\alpha_i$ which converges a.s. Hence, AA exists.

(ii) Let $u = Ax$. Then $\sum_{i=1}^{\infty}(u, e_i)Be_i = \left(\sum_{i=1}^{\infty} \alpha_i x_i\right)\sum_{i=1}^{\infty} a_i\alpha_i e_i$ which converges a.s. since $(a_i) \in l_s$. Hence, BA exists.

(iii) Let $c \in l_s$ but $c \notin l_{p/2}$. Put $u = Bc$. Then we have

$$\sum_{i=1}^{\infty}(u, e_i)Be_i = \sum_{i=1}^{\infty} \alpha_i^2 c_i e_i.$$

Because $c = (c_i) \notin l_{p/2}$ and $2s > p$, by Lemma 3.5,

$$\sum_{i=1}^{\infty} |\alpha_i^2 c_i|^s = \sum_{i=1}^{\infty} |\alpha_i\sqrt{|c_i|}|^{2s} = \infty.$$

Hence, BB does not exist.

(iv) Let $c = (c_i) \in l_s$ and $c \notin l_{p/2}$ $(c_i > 0,\ i = 1, 2, \ldots)$. Put $u = Bc$. Then we have

$$\sum_{i=1}^{\infty}(u, e_i)Ae_i = a\sum_{i=1}^{\infty} \alpha_i^2 c_i.$$

Because $c = (c_i) \notin l_{p/2}$, by Lemma 3.5, we have $\sum_{i=1}^{\infty} \alpha_i^2 c_i = \sum_{i=1}^{\infty} |\alpha_i \sqrt{c_i}|^2 = \infty$. Hence, AB does not exist.

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