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On Nonlinear n-widths and n-term Approximation

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Abstract. In the present paper, we introduce definitions of nonlinear n-widths based on the best n-term approximations by families Φ of functions. The asymptotic degrees of the n-widths of Besov classes of periodic functions are given. Moreover, we show that the family Φ, constituted from de la Vallée Pousin kernels, are asymptotically optimal for these n-widths.

1. Introduction

Let X be a normed linear space and $\Phi := \{\varphi_1, \varphi_2, ..., \varphi_k, ...\}$ a family of elements in X. We are interested in the n-term approximation of an element $f \in X$ by the linear combinations φ of elements from Φ of the form

$$\varphi = \sum_{j=1}^{n} a_{k_j} \varphi_{k_j}, \quad \varphi_{k_j} \in \Phi.$$

Denote by $M_n(\Phi)$ the set of all these linear combinations. The set $M_n(\Phi)$ is a nonlinear continuous manifold in X in the following sense. Let l_∞ be the normed linear space of all bounded sequences of numbers $x = \{x_k\}_{k=1}^{\infty}$, equipped by the norm

$$\|x\|_\infty := \sup_{1 \le k < \infty} |x_k|,$$

and M_n the subset in l_∞ of all $x = \{x_k\}_{k=1}^{\infty}$ with at most n $x_k \neq 0$. Consider the mapping R_Φ from the metric space M_n into X defined by

$$R_\Phi(x) := \sum_{j=1}^{n} x_{k_j} \varphi_{k_j},$$

for $x = \{x_k\}_{k=1}^{\infty}$ with $x_k = 0$ for $k \neq k_1, ..., k_n$. From the definitions, we have

Lemma 1. *If the family Φ is bounded, i.e., $\|\varphi_k\| \leq c$ for $k = 1, 2, \ldots$, then R_Φ is a continuous mapping from M_n into X and $M_n(\Phi) = R_\Phi(M_n)$.*

This lemma shows that $M_n(\Phi)$ is the image of the set M_n under the continuous mapping R_Φ, i.e., a nonlinear manifold in X, parameterized continuously in the metric of l_∞ by M_n.

Given $f \in X$, we define

$$\sigma_n(f, \Phi, X) := \inf_{\varphi \in M_n(\Phi)} \|f - \varphi\|. \tag{1}$$

Next, if $W \subset X$, we put

$$\sigma_n(W, \Phi, X) := \sup_{f \in W} \sigma_n(f, \Phi, X). \tag{2}$$

The quantities (1) and (2) are called the best n-term approximations by the family Φ of f and W, respectively. There has recently been great interest in the best n-term approximation by various families of functions in both theoretical and application aspects. There were several works on the best n-term approximation, among which [4, 11] were on the best n-term approximation by B-splines, [2, 3] on the best n-term approximation by wavelets and [6, 9] on the best n-term approximation by trigonometric functions (exponents), etc.

The nonlinear manifolds $M_n(\Phi)$ in the definitions (1) and (2) are too general (even for bounded Φ) to make them useful. Thus, if X is separable and Φ is dense in the unit ball of X, then $\sigma_n(f, \Phi, X) = 0$ for any $f \in X$. Therefore, the first problem which actually arises is to impose reasonable conditions on Φ and methods of approximation by the elements of $M_n(\Phi)$. One of the approaches to dealing with this problem is to restrict approximations by the elements of $M_n(\Phi)$ with only continuous methods which are represented as continuous mappings from W into $M_n(\Phi)$. This approach certainly leads to a notion of nonlinear width. We will consider this problem in detail in the next section. In particular, we introduce two definitions of nonlinear n-widths $\alpha_n(W, X)$ and $\beta_n(W, X)$ based on the n-term approximation (2). The authors of [1] have suggested a notion of nonlinear manifold n-width $\delta_n(W, X)$ based on nonlinear approximations by nonlinear manifolds continuously parameterized by $\boldsymbol{R}^n$. The other approach is to impose on the family Φ "minimality properties" [9]. This approach will be discussed in Sec. 5. As an auxiliary part of the present paper, Sec. 3 is devoted to the nonlinear n-widths α_n and β_n of finite-dimensional sets. In Sec. 4, we prove the asymptotic degrees of the nonlinear n-widths α_n and β_n of Besov classes of multivariate periodic functions.

2. Nonlinear Widths

Let W be a subset in the normed linear space X. The nonlinear n-width $\alpha_n(W, X)$ is defined by

$$\alpha_n(W, X) := \inf_{\Phi, F} \sup_{f \in W} \|f - R_\Phi(F(f))\|,$$

where the infimum is taken over all continuous mappings F from W into M_n and all bounded families Φ in X. The n-width $\alpha_n(W, X)$ expresses the error of the best continuous method of n-term approximation by Φ of the elements in W, i.e., by the elements from nonlinear manifolds $M_n(\Phi)$. We next introduce another nonlinear n-width as a characterization of the best continuous method of nonlinear approximations by the images of all continuous mappings from M_n into X, namely, the nonlinear n-width $\beta_n(W, X)$ is defined by

$$\beta_n(W, X) := \inf_{R,F} \sup_{f \in W} \|f - R(F(f))\|,$$

where the infimum is taken over all continuous mappings F from W into M_n and R from M_n into X. There are other notions of nonlinear width (see, e.g., [3, 14] for details). We would like to recall among them the well-known Alexandroff n-width $a_n(W, X)$ and the nonlinear manifold n-width $\delta_n(W, X)$ [1], which are more closely related to the n-widths $\alpha_n(W, X)$ and $\beta_n(W, X)$ and have a more explicit approximative meaning. The Alexandroff n-width $a_n(W, X)$ is defined by

$$a_n(W, X) := \inf_{F,K} \sup_{f \in W} \|f - F(f)\|,$$

where the infimum is taken over all compact subsets $K \subset X$ of topological dimensions $\leq n$ and all continuous mappings F from W into K (see, e.g., [3, 8] for the definition of topological dimension). The nonlinear manifold n-width $\delta_n(W, X)$ is defined by

$$\delta_n(W, X) := \inf_{R,F} \sup_{f \in W} \|f - R(F(f))\|,$$

where the infimum is taken over all continuous mappings F from W into $\boldsymbol{R}^n$ and R from $\boldsymbol{R}^n$ into X.

Lemma 2. *Let W be a compact subset in the normed linear space X. Then the following inequalities hold*

$$a_n(W, X) \leq \beta_n(W, X) \leq \alpha_n(W, X), \tag{3}$$

$$\delta_{2n+1}(W, X) \leq a_n(W, X) \leq \beta_n(W, X) \leq \delta_n(W, X). \tag{4}$$

Proof. The inequalities $\delta_{2n+1}(W, X) \leq a_n(W, X) \leq \delta_n(W, X)$ in (4) were proved in [8]. The inequalities $a_n(W, X) \leq \beta_n(W, X)$ in (3) and $\beta_n(W, X) \leq \delta_n(W, X)$ in (4) directly follow from the definitions. To complete the proof of the lemma, we will check the inequality $a_n(W, X) \leq \beta_n(W, X)$ in (3). Note that if $K \subset M_n$ is a compact subset, then the topological dimensions of K are not larger than n. This means that for arbitrary $\varepsilon > 0$, there exists a finite open ε-covering of multiplicity $\leq n + 1$. This property is implied from the fact that for arbitrary $\varepsilon > 0$, there exists an (infinite) open ε-covering of multiplicity $n + 1$ for M_n. Hence, the inequality $a_n(W, X) \leq \beta_n(W, X)$ can be proved in a way similar to the proof of the inequality $a_n(W, X) \leq \delta_n(W, X)$ which was given in [8]. ■

In this and the following sections, we denote by γ_n either α_n or β_n.

Lemma 3. *Let the linear space L be normed by two equivalent norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, and let W be a subset of L. Assume W is compact in these norms, and $\gamma_s(W,X) > 0$. Then we have*

$$\gamma_{n+s}(W,Y) \le \gamma_n(BX,Y)\gamma_s(W,X),$$

where $BX := \{x \in L\colon \|x\|_X \le 1\}$.

Proof. The proofs of this lemma are similar for α_n and β_n. We prove it, for example, for α_n. Since the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ are equivalent, the boundedness and continuity properties are understood with respect to the topology of these norms. It is convenient to represent a family $\Phi = \{\varphi_1, \varphi_2, ..., \varphi_k, ...\}$ in the form $\Phi = \{\varphi_k\}_{k\in K}$ and l_∞ as the linear normed space $l_\infty(K)$ of all bounded sequences $x = \{x_k\}_{k\in K}$ with the usual supremum norm, where K is an abstract accounting set of indices. Let $F_1 : W \longrightarrow M_s$ and $F_2 : BX \longrightarrow M_n$ be any continuous mappings, and $\Phi = \{\varphi_k\}_{k\in Q_1}$ and $\Phi_2 = \{\varphi_k\}_{k\in Q_2}$ are any bounded families, where Q_1 and Q_2 are accounting sets of indices and $Q_1 \cap Q_2 = \emptyset$. Put $G(f) := f - R_{\Phi_1}(F_1(f))$ and

$$\delta := \sup_{f\in W} \|G(f)\|_X.$$

Since W is compact and $\alpha_s(W,X) > 0$, we have $0 < \delta < \infty$. We define the family $\Phi := \{\varphi_k\}_{k\in Q}$, where $Q := Q_1 \cup Q_2$, and the mapping $F : W \longrightarrow M_{n+s}$ as follows. Consider M_s, M_n and M_{n+s} as subsets in $l_\infty(Q_1), l_\infty(Q_2)$ and $l_\infty(Q)$, respectively. Then the set $M^*_{n+s} := \{(x,y) : x \in M_s,\ y \in M_n\}$ can be represented as a subset of M_{n+s}. Let the mapping $F : W \longrightarrow M^*_{n+s}$ be given by

$$F(f) := (F_1(f), \delta F_2(G(f)/\delta)),\ f \in W.$$

It is easily seen that Φ is bounded and F is a continuous mapping from W into M_{n+s}. We have $G(f)/\delta \in BX$ and

$$f - R_\Phi(F(f)) = \delta\{(G(f)/\delta - R_{\Phi_2}(F_2(G(f)/\delta))\}.$$

Hence,

$$\sup_{f\in W} \|f - R_\Phi(F(f))\|_Y \le \delta \sup_{f\in BX} \|f - R_{\Phi_2}(F_2(f))\|_Y.$$

This proves the lemma for α_n. ∎

3. Nonlinear Widths of Finite-dimensional Sets

Let us consider l_∞ as a linear space. For $0 < p \le \infty$, denote by l^m_p the linear subspace of all $x = \{x_k\}_{k=1}^\infty$ with $x_k = 0$ for $k = m+1, m+2, ...,$ equipped by the norm

$$\|\{x_k\}\|_{l^m_p} = \|x\|_{l^m_p} := \left(\sum_{k=1}^m |x_k|^p\right)^{1/p} \quad (5)$$

with a change to the max norm when $p = \infty$. If $m \geq n$, let M_n^m be the subset in M_n of all $x = \{x_k\}_{k=1}^\infty$ with $x_k = 0$ for $k = m+1, m+2, ...$. Note that the metrics of l_∞ and l_p^m generate the same topology in the linear subspace of all elements $x = \{x_k\}_{k=1}^\infty$ with $x_k = 0$ for $k = m+1, m+2, ...$, particularly, in M_n^m. Moreover, every continuous mapping S from the normed linear space l_p^m into the metric space $M_n^m \subset l_\infty$ can be considered as the supercomposition $S = R_E \circ F_S$ of the continuous mapping F_S from l_p^m into M_n and R_E, where $F_S(x) = S(x)$ for $x \in M_n^m$ and $E = \{e^1, e^2, ..., e^j, ...\}$, e^j is the jth basic vector in l_∞, i.e., $e_k^j = 1$ for $j = k$ and $e_k^j = 0$ for $j \neq k$. It is sometimes convenient to represent l_p^m as the space of finite sequences $x = \{x_k\}_{k=1}^m$, equipped by the norm (5).

Denote by B_p^m the unit ball in l_p^m.

Lemma 4. *Let $0 < p \leq \infty$, $1 \leq q \leq \infty$ and $m > n$. Then we have*

$$\gamma_n(B_p^m, l_q^m) = A_{p,q}(m,n),$$

where

$$A_{p,q}(m,n) = \begin{cases} (n+1)^{1/q-1/p}, & \text{for } p < q \\ 1, & \text{for } p = q \\ (m-n)^{1/q-1/p}, & \text{for } p > q. \end{cases}$$

In addition, we can explicitly construct a continuous mapping $S : l_p^m \longrightarrow M_n^m$ such that $S = R_E \circ F_S$, $S(\lambda x) = \lambda S(x)$ for $\lambda \geq 0$, and

$$\sup_{x \in B_p^m} \|x - R_E(F_S(x))\|_{l_q^m} \leq A_{p,q}(m,n). \tag{6}$$

Proof. This lemma can be proved in a similar way to that of [8, Lemma 2.5]. For completeness of the present paper and understanding further discussions, we prove it here. We first construct a continuous mapping $S : l_p^m \longrightarrow M_n^m$ such that $S = R_E \circ F_S$, $S(\lambda x) = \lambda S(x)$ for $\lambda \geq 0$, and the inequality (6) holds. For the case $p \geq q$, the mapping S is defined as the linear projector

$$S(x) = \{x_1, x_2, ..., x_n, 0, ..., 0, ...\}, \text{ for } x = \{x_k\}_{k=1}^\infty \in l_p^m.$$

From the Hölder inequality, it is easy to check that

$$\|x - S(x)\|_{l_q^m} = \left(\sum_{k=n+1}^m |x_k|^q\right)^{1/q} \leq A_{p,q}(m,n)\|x\|_{l_p^m}. \tag{7}$$

Next, we consider the case $p < q$. In order to define S, we use an idea in [12] for establishing the upper bound of $a_n(B_p^m, l_q^m)$. If $x \in l_p^m$ and

$$|x_{k_1}| \geq |x_{k_2}| \geq \cdots \geq |x_{k_m}|,$$

we put $S(x) = y$, where

$$y_{k_j} = \begin{cases} x_{k_j} - |x_{k_{n+1}}| \operatorname{sign} x_{k_j}, & \text{for } k = 1, ..., n \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\|x - S(x)\|_{l_q^m} \le A_{p,q}(m,n)\|x\|_{l_p^m}. \tag{8}$$

This inequality was proved in [12] for the case $1 \le p \le \infty$ (see also [14]). The case $0 < p < 1$ can be treated similarly. It is easily seen that S is a continuous mapping from l_p^m into M_n^m and $S(\lambda x) = \lambda S(x)$, $\lambda \ge 0$, for both the cases $p \ge q$ and $p < q$. From (7) and (8), we obtain (6) and the following upper bound

$$\gamma_n(B_p^m, l_q^m) \le A_{p,q}(m,n). \tag{9}$$

To prove the lower bound

$$\gamma_n(B_p^m, l_q^m) \ge A_{p,q}(m,n), \tag{10}$$

we put $X = l_p^m$, $Y = l_q^m$, $W = B_q^m$, $s = m - n - 1$, and apply Lemma 3. We have

$$\gamma_{m-1}(B_q^m, l_q^m) \le \gamma_n(B_p^m, l_q^m)\gamma_{m-n-1}(B_q^m, l_p^m). \tag{11}$$

The inequality (9) gives

$$\gamma_{m-n-1}(B_q^m, l_p^m) \le A_{q,p}(m, m-n-1). \tag{12}$$

On the other hand, the inequality (3) and the equality $a_{m-1}(B_q^m, l_q^m) = 1$ [12](see also [8]) imply

$$\gamma_{m-1}(B_q^m, l_q^m) \ge 1.$$

This and (11) and (12) prove (10). ■

4. Nonlinear Widths of Besov Classes

If $\alpha > 0$ and $0 < p, \theta \le \infty$, denote by $B_{p,\theta}^\alpha$ the Besov space of all functions f defined on the d-dimensional torus $\boldsymbol{T}^d := [0, 2\pi]^d$, such that the norm

$$\|f\|_{B_{p,\theta}^\alpha} := \|f\|_p + |f|_{B_{p,\theta}^\alpha}$$

is finite, where

$$|f|_{B_{p,\theta}^\alpha} := \left(\int_0^\infty \{t^{-\alpha}\omega^r(f,t)_p\}^\theta dt/t\right)^{1/\theta}, \theta < \infty,$$

with the usual change to the supremum when $\theta = \infty$, r is any natural number larger than α, and $\omega^r(f,t)_p$ is the p-integral modulus of smoothness of order r of the function f. Here, $\|\cdot\|_p$ denotes the p-integral norm of $L_p(\boldsymbol{T}^d)$.

In what follows, the notation $A \ll A'$ means $A \le cA'$ with absolute constant c, and the notation of asymptotic equivalence $A \asymp A'$ means $A \ll A'$ and $A' \ll A$. Denote by $K_{p,\theta}^\alpha$ the unit ball in $B_{p,\theta}^\alpha$.

Theorem 1. *Let* $1 \le p, q \le \infty$, $0 < \theta \le \infty$ *and* $\alpha > \max\{0, d/p - d/q\}$. *Then we have*

$$\gamma_n(K_{p,\theta}^\alpha, L_q(\boldsymbol{T}^d)) \asymp n^{-\alpha/d}.$$

Proof. The lower bound

$$\gamma_n(K^\alpha_{p,\theta}, L_q(\boldsymbol{T}^d)) \gg n^{-\alpha/d}$$

follows from the inequality (3) and [8, Theorem 3.2]. To prove the upper bound

$$\gamma_n(K^\alpha_{p,\theta}, L_q(\boldsymbol{T}^d)) \ll n^{-\alpha/d}, \tag{13}$$

it is sufficient to construct a continuous mapping $F : K^\alpha_{p,\theta} \longrightarrow M_n$ and a family $\Phi \subset L_q(\boldsymbol{T}^d)$ such that

$$\sup_{x \in K^\alpha_{p,\theta}} \|f - R_\Phi(F(f))\|_q \ll n^{-\alpha/d}. \tag{14}$$

For the sake of simplicity, we are restricted to prove the case $d = 1$. The case $d > 1$ can be treated similarly. Moreover, since $B^\alpha_{p,\theta} \subset H^\alpha_p := B^\alpha_{p,\infty}$, it is enough to verify (14) for the case of $K^\alpha_p := K^\alpha_{p,\infty}$. For the nonnegative integer ν, let

$$V_\nu(x) := \frac{1}{2} + \sum_{k=1}^{\nu} \cos kx + \sum_{k=\nu+1}^{2\nu} \frac{2\nu - k}{\nu} \cos kx = \frac{\sin(\nu x)\sin(3\nu x/2)}{2\nu \sin^2(x/2)}$$

be the de la Vallée Pousin kernel of order ν. For functions $f \in L_q(\boldsymbol{T})$, the convolution

$$V_\nu f := f * V_\nu$$

defines the de la Vallée Pousin sum of f. Note that $V_\nu f \in \mathcal{T}_{2\nu-1}$, where $\mathcal{T}_m$ denotes the space of all trigonometric polynomials of order $\le m$. Next, we put

$$v_0 f := V_1 f;\ v_k f := V_{2^k} f - V_{2^{k-1}} f,\ k = 1, 2, \ldots.$$

If $\alpha > 0, 1 \le p \le \infty$, then [13]

$$\|f\|_{H^\alpha_p} \asymp \sup_{0 \le k < \infty} 2^{\alpha k} \|v_k f\|_p. \tag{15}$$

Given a function f defined on $\boldsymbol{T}$, we set

$$S_\nu f(x) := \sum_{k=0}^{3\nu-1} f(hk) S_\nu(x - hk),$$

where

$$S_\nu(x) := (3\nu)^{-1} V_\nu(x);\ h := 2\pi(3\nu)^{-1}.$$

It is easy to check that

$$f = S_\nu f \text{ for every } f \in \mathcal{T}_\nu. \tag{16}$$

Let T^ν_p be the subspace of $f \in L_p(\boldsymbol{T})$, spanned on $\Phi_\nu := \{S_\nu(\cdot - hk)\}_{k=0}^{3\nu-1}$. If

$$f(x) := \sum_{k=0}^{3\nu-1} c_k S_\nu(x - hk) \tag{17}$$

is a function in T_p^ν, then

$$f(hk) = c_k,\ k = 0, 1, ..., 3\nu - 1, \tag{18}$$

and moreover [7],

$$\|f\|_p \asymp \nu^{-1/p}\|\{f(hk)\}\|_{l_p^{3\nu}},\ 1 \le p \le \infty. \tag{19}$$

Note that the correspondence of $f \in T_p^\nu$ with $\{f(hk)\}_{k=0}^{3\nu-1}$ forms an isomorphism from T_p^ν onto $l_p^{3\nu}$. Denote it by J. Let S_p^ν be the unit ball in T_p^ν and $F_S : l_p^{3\nu} \longrightarrow M_n$ the continuous mapping defined in Lemma 4 for $m = 3\nu$, and $F^* = F_S \circ J$. Obviously, F^* is a continuous mapping from T_p^ν into M_n. By Lemma 4 and (19), we have for $3\nu > m$

$$\sup_{f \in S_p^\nu} \|f - R_{\Phi_\nu}(F^*(f))\|_q \ll \nu^{1/p-1/q} A_{p,q}(3\nu, n), \tag{20}$$

and $(R_{\Phi_\nu} \circ F^*)(\lambda f) = \lambda(R_{\Phi_\nu} \circ F^*)(f)$ for $\lambda \ge 0$. We will consider the case $p < q$ (the case $p \ge q$ can be treated similarly with a slight modification). Given a natural number $n > 4$, we find the nonnegative integer s by the condition $2^{s+2} \le n < 2^{s+3}$. Let ε be a fixed number satisfying the inequalities $0 < \varepsilon < (\alpha - \beta)/\beta$, where $\beta = 1/p - 1/q > 0$. We put

$$n_s = 2^{s+1};\ n_k = [an2^{-\varepsilon(k-s)}],\ k = s+1, s+2, ...$$

with the parameter a chosen such that

$$\sum_{k=s}^{\infty} n_k \le n. \tag{21}$$

If $f \in K_p^\alpha$, then the inequality $\|V_\nu f\|_p \le 3\|f\|_p$, for any nonnegative integer ν, and (15) give

$$f = V_{2^{s-1}} f + \sum_{k=s}^{\infty} v_k f = \sum_{k=s}^{\infty} f_k, \tag{22}$$

the series converging in the $L_p(\boldsymbol{T})$-norm, and

$$\|f_s\|_p \le \lambda;\ \|f_k\|_p \le \lambda 2^{-\alpha k},\ k = s+1, s+2, ..., \tag{23}$$

where λ is an absolute constant and $f_s = V_{2^{s-1}} f;\ f_k = v_{k-1} f,\ k = s+1, s+2, ...$ By virtue of (16), $f_k \in \mathcal{T}_{2^k}$, $k = s, s+1, ...$, and therefore, by (23),

$$f_s \in \lambda S_p^{2^s};\ f_k \in \lambda 2^{-\alpha k} S_p^{2^k},\ k = s+1, s+2, ... \tag{24}$$

Put

$$\Phi(k) = \Phi_{2^k} := \{\varphi_m^k\}_{m \in Q_k},\ k = s, s+1, ...,$$

where $\varphi_m^k := S_{2^k}(\cdot - h_k m)$, $h_k = (2\pi/3)2^{-k}$, and $Q_k := \{0, 1, ..., 3.2^k - 1\}$. Let

$$F_k^* : T_p^{2^k} \longrightarrow M_{n_k},\ k = s+1, s+2, ...,$$

the continuous mapping defined by $F_k^* := F_S \circ J$ for $\nu = 2^k$. From (20) and (24), we obtain

$$\sup_{f \in K_p^\alpha} \|f_k - R_{\Phi(k)}(F_k^*(f_k))\|_q \ll 2^{-(\alpha-\beta)k} A_{p,q}(3.2^k, n_k), \ k = s+1, s+2, \ldots . \quad (25)$$

Further, we let $F_s^* = J$, where the isomorphism J is defined above for $\nu = 2^s$. By (16), we have $f = S(f) = (R_{\Phi_\nu} \circ J)(f)$ for every $f \in T_p^\nu$, $\nu = 2^s$. Hence,

$$\sup_{f \in K_p^\alpha} \|f_s - R_{\Phi(s)}(F_s^*(f_s))\|_q = 0. \quad (26)$$

Let

$$\Phi := \bigcup_{k=s}^{\infty} \Phi(k), \quad (27)$$

and

$$F(f) := \{F_k^*(f_k)\}_{k=s}^{\infty} \ \text{ for } f = \sum_{k=s}^{\infty} f_k \in K_p^\alpha. \quad (28)$$

Consider l_∞ as $l_\infty(Q)$ with $Q := \{(k, m_k) : k = s, s+1, \ldots, m_k \in Q_k\}$ (see the proof of Lemma 3). Then, by (21) and (22), F is a continuous mapping from K_p^α into M_n. Using (25) and (26) and the inequalities $0 < \varepsilon < (\alpha - \beta)/\beta$, $\beta > 0$ and $2^{s+2} \le n < 2^{s+3}$, by simple computation, we get, for any $f \in K_p^\alpha$,

$$\|f - R_\Phi(F(f))\|_q \le \sum_{k=s}^{\infty} \|f_k - R_{\Phi(k)}(F_k^*(f_k))\|_q$$

$$\ll \sum_{k=s+1}^{\infty} 2^{-(\alpha-\beta)k} A_{p,q}(3.2^k, n_k) \ll n^{-\alpha}.$$

Thus, (14) has been proved for Φ and F which are defined in (27) and (28). ∎

Let U_p^α be the unit ball of the Sobolev space W_p^α (see, e.g., [10] for the definition). Theorem 1 and the well-known embeddings between W_p^α and $B_{p,\theta}^\alpha$ imply

Corollary 1. *Let* $1 \le p, q \le \infty$ *and* $\alpha > \max\{0, d/p - d/q\}$. *Then we have*

$$\gamma_n(U_p^\alpha, L_q(\boldsymbol{T}^d)) \asymp n^{-\alpha/d}.$$

5. Sufficient Conditions for the Lower Bound

In this section, we discuss sufficient properties of the family Φ, which would give a reasonable sense for the quantity $\sigma_n(W, \Phi, L_q(G))$ for well-known classes W of functions defined in G, where G is either $\boldsymbol{T}^d$ or a bounded domain in $\boldsymbol{R}^d$. This problem was studied in [8]. One of the restrictions on Φ suggested in [9] is the linear independence of Φ. However, many important families Φ such as wavelets, B-splines and de la Vallée Pousin kernels do not have this property. We would like to replace it by a weaker one. Namely, let $\Phi := \{\varphi_1, \varphi_2, \ldots, \varphi_k, \ldots\}$ be a family of functions on $L_q(G)$. We require that there exists a sequence $\{\Phi(k)\}_{k=1}^{\infty}$ of subsets of Φ with $\mathrm{card}\Phi(k) \asymp 2^{dk}$, satisfying the following conditions:

(i) $\Phi(k)$ is linearly independent for $k = 1, 2, \ldots$;

(ii) For $1 \le q \le \infty$, the following norms equivalence holds:

$$\Big\| \sum_{\varphi \in \Phi(k)} a_\varphi \varphi \Big\|_q \asymp \Big(2^{-dk} \sum_{\varphi \in \Phi(k)} |a_\varphi|^q \Big)^{1/q}$$

with the change to the usual supremum when $q = \infty$;

(iii) There exists a nonnegative integer k_0 such that for $k = 1, 2, \ldots,\ n = 1, 2, \ldots$, and $1 \le q \le \infty$ and for any $\varphi \in \operatorname{span}\Phi(k)$, the following inequality holds:

$$\sigma_n(\varphi, \Phi(k + k_0), L_q(G)) \ll \sigma_n(\varphi, \Phi, L_q(G));$$

(iv) For $\alpha > 0$, the following inequality holds for any $\varphi \in \operatorname{span}\Phi(k)$

$$\|\varphi\|_{B^\alpha_{\infty,1}(G)} \ll 2^{\alpha k} \|\varphi\|_{L_\infty(G)}.$$

Denote by $K^\alpha_{p,\theta}(G)$ the unit ball of the Besov space $B^\alpha_{p,\theta}(G)$ of functions defined on G.

Theorem 2. *Let $1 \le p, q \le \infty$, $0 < \theta \le \infty$ and let the family Φ satisfy the conditions (i)–(iv). Then we have*

$$\sigma_n(K^\alpha_{p,\theta}(G), \Phi, L_q(G)) \gg n^{-\alpha/d}. \tag{29}$$

Proof. It is easily seen that it is sufficient to prove the theorem for the case $p = \infty$, $q = \theta = 1$. Similar to the proof of [9, Lemma 1] from the properties (i)–(iii), one can verify the following inequality:

$$\sigma_n(\Phi(k)_\infty, \Phi, L_q(G)) \gg 1, \tag{30}$$

where $\Phi(k)_\infty$ denotes the set of all $\varphi \in \operatorname{span}\Phi(k)$ such that $\|\varphi\|_{L_\infty(G)} \le 1$. The condition (iv) implies $\Phi(k)_\infty \subset \lambda 2^{\alpha k} K^\alpha_{\infty,1}(G)$ for some absolute constant λ. Hence, by (30), we obtain (29) for the case $p = \infty$, $q = \theta = 1$. ■

Corollary 2. *Let $1 \le p, q \le \infty$, $0 < \theta \le \infty$ and $\alpha > \max\{0, d/p - d/q\}$, and let Φ be the family of de la Vallée Pousin kernels defined in (27). Then we have*

$$\sigma_n(K^\alpha_{p,\theta}, \Phi, L_q(T)) \asymp n^{-\alpha}. \tag{31}$$

$$\sigma_n(U^\alpha_p, \Phi, L_q(T)) \asymp n^{-\alpha}. \tag{32}$$

Proof. It is easy to check that Φ is a family satisfying the conditions (i)–(iv). Hence, by Theorems 1 and 2 and Corollary 1, we obtain (31) and (32). ∎

A multivariate generalization is also valid for the class $K^{\alpha}_{p,\theta}$ of functions defined in $\boldsymbol{T}^d$ and the space $L_q(\boldsymbol{T}^d)$. Condition (ii) could be replaced by the following condition considered in [9]:

(ii′) There exists a finite subset $G_k \subset G$ for $k = 1, 2, \ldots$ such that $\mathrm{card}G_k \ll 2^{dk}$, and for any $\varphi \in \mathrm{span}\Phi(k)$ and $1 \le q \le \infty$, the following norms equivalence holds:

$$\left((\mathrm{card}G_k)^{-1} \sum_{x \in G_k} |\varphi(x)|^q \right)^{1/q} \asymp \|\varphi\|_q$$

with a change to the usual supremum when $q = \infty$.

Minimality properties of Φ, including the linear independence of Φ, conditions (ii′) and (iii), and "the Bernstein inequality", were formulated for establishing the lower bound [9]

$$\sigma_n(U^{\alpha}_{\infty}(G), \Phi, L_1(G)) \gg n^{-\alpha/d},$$

where $U^{\alpha}_p(G)$ denotes the unit ball of the Sobolev space $W^{\alpha}_p(G)$ of functions on G. Theorem 2(i) substitutes the condition of linear independence of Φ, while condition (iv) is a modification of "the Bernstein inequality". Corollary 2 particularly shows that, for the case $p < q$, the family Φ constituted from de la Vallée Pousin kernels defined in (27) gives the asymptotic degree of $\sigma_n(W, \Phi, L_q(\boldsymbol{T}))$ for W, the Besov class $K^{\alpha}_{p,\theta}$ or Sobolev class U^{α}_p, better than the family of trigonometric exponents $\{e^{ikx}\}_{k=0}^{\infty}$ considered in [6, 9].

Remark. We can consider nonperiodic analogs of Theorem 1 and Corollaries 1 and 2 for the Besov class $B^{\alpha}_{p,\theta}(\boldsymbol{I}^d)$ and Sobolev class $W^{\alpha}_p(\boldsymbol{I}^d)$ of functions defined on $\boldsymbol{I}^d := [0, 1]^d$. Let $\psi(x) := N(x_1) \cdots N(x_d)$ be the tensor product B-spline where $N(t) := N(t; 0, \ldots, r)$ is the univariate B-spline of order r with knots at the points $0, 1, \ldots, r$ and r is a fixed natural number with $r > \alpha$. Let $\psi_{k,m} := \psi(2^k \cdot - m)$, $m \in \boldsymbol{Z}^d$, $k \in \boldsymbol{Z}$ be the translated dilates of ψ. For n-term approximation of the functions from $B^{\alpha}_{p,\theta}(\boldsymbol{I}^d)$ and $W^{\alpha}_p(\boldsymbol{I}^d)$, we take the family of algebraic polynomials and wavelets:

$$\Psi := \left\{ \{\psi_m\}_{m \in Q_r}, \{\psi_{k,m}\}_{0 \le k < \infty, m \in \Lambda_k} \right\},$$

where $\psi_m(x) := x_1^{m_1} \cdots x_d^{m_d}$; $Q_r := \{m \in \boldsymbol{Z}^d : 0 \le m_j < r, j = 1, \ldots, d\}$. Λ_k is the set of those indices $m \in \boldsymbol{Z}^d$ such that $\psi(2^k \cdot - m)$ does not vanish identically on $\boldsymbol{I}^d$. It was proved in [5] that the functions f from $B^{\alpha}_{p,\theta}(\boldsymbol{I}^d)$ have a wavelet decomposition into ψ_m and $\psi_{k,m}$ with the coefficient functionals depending continuously on f and with the corresponding quasinorms equivalence. Using this wavelet decomposition, in a way similar to the proofs of Theorem 1 and Corollaries 1 and 2, we can prove the following. Let $1 \le q \le \infty$, $0 < p, \theta \le \infty$ and $\alpha > \max\{0, d/p - d/q\}$. Then we have

$$\gamma_n(K^{\alpha}_{p,\theta}(\boldsymbol{I}^d), L_q(\boldsymbol{I}^d)) \asymp \gamma_n(U^{\alpha}_p(\boldsymbol{I}^d), L_q(\boldsymbol{I}^d)) \asymp n^{-\alpha/d},$$

$$\sigma_n(K^{\alpha}_{p,\theta}(\boldsymbol{I}^d), \Psi, L_q(\boldsymbol{I}^d)) \asymp \sigma_n(U^{\alpha}_p(\boldsymbol{I}^d), \Psi, L_q(\boldsymbol{I}^d)) \asymp n^{-\alpha/d}.$$

We can construct a method of n-term approximation of the functions from $K^{\alpha}_{p,\theta}(I^d)$ by the family Ψ. This method is completely similar to the method of n-term approximation by de la Vallée Poussin kernels in the proof of Theorem 1. Another method of n-term approximation by Ψ was treated in [3].

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