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Short Communication

Morita Invariance of Entire Current Cyclic Homology

Nguyen Van Thu

Institute of Mathematics, P.O. Box 631, Bo Ho 10.000, Hanoi, Vietnam

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This note is a continuation of [6]. Our main technical point is to use the Cuntz–Quillen theory [2, 3] of non-commutative differential forms over algebras. The main result of this note is the Morita invariance of the entire cyclic current homology.

Let A be an involutive Banach algebra and $\{A_\lambda\}_{\lambda\in I}$ the family of ideals of A with trace, i.e., with a map $\tau_\lambda : A_\lambda \to \mathbf{C}$, satisfying the following four conditions:

(1) τ_λ is a continuous linear map, $\|\tau_\lambda\| = 1$;
(2) $\tau_\lambda(a^*a) \geq 0$;
(3) $\tau_\lambda(a^*a) = 0$ for every $\lambda \in I$, if and only if $a = 0$;
(4) τ_λ is ad$A$-invariant, i.e., $\tau_\lambda(xa) = \tau_\lambda(ax)$, $\forall x \in A$ and $\forall a \in A_\lambda$.

For every λ, the map, therefore, defines a scalar product over A_λ by the formula

$$\langle a, b\rangle_{\tau_\lambda} := \tau_\lambda(a^*b).$$

In [6], we constructed the periodic bicomplex of entire chains $C_e^0(A)$ and reduced (B^*, b^*)-bicomplex $B_e^0(\widetilde{A})_{\text{red}}$. The entire current cyclic homology $HE_*(A)$ of involutive Banach algebra A is the homology of the above bicomplex $C_e^0(A)$ (or $B_e^0(\widetilde{A})_{\text{red}}$) of entire current chains. We have the *total complex* Tot $(C_e^0(A))$ where, by definition,

$$\text{Tot}\,(C^0(A))^{\text{ev}} := \text{Tot}\,(C^0(A))^{\text{odd}} := \oplus_{n\geq 0} C_n^0(A),$$

with

$$C_n^0(A) := \text{Hom}\,(\varinjlim C_e^n(\overline{A}_\lambda, \tau_\lambda), \mathbf{C}).$$

For a positive integer q, $M_q(\overline{A}_\lambda) = M_q(\mathbf{C}) \otimes \overline{A}_\lambda$ is the algebra of $q \times q$ matrices over $\overline{A}_\lambda$. The map $\text{Tr}_\lambda : C^n(A_\lambda, \tau_\lambda) \to C^n(M_q(\overline{A}_\lambda), \tau_\lambda)$, defined by

$$(\text{Tr}_\lambda \phi_\lambda^n)(m^0 \otimes a_\lambda^0, ..., m^n \otimes a_\lambda^n) = \text{tr}(m^0...m^n)\phi_\lambda^n(a_\lambda^0, ..., a_\lambda^n)$$

is called a *trace map*, where $\mathrm{tr} : M_q(\mathbf{C}) \to \mathbf{C}$ is the usual trace of matrices.

It is easy to check that the homomorphism Tr_λ can be extended to the corresponding operator, denoted by $\mathrm{Tr}_\lambda : \varinjlim C^n(\overline{A}_\lambda, \tau_\lambda) \to \varinjlim C^n(M_q(\overline{A}_\lambda), \tau_\lambda)$, for all fixed n. We can define the continuous homomorphism Tr^*, which is just the adjoint of Tr_λ. In [6], we have $\mathrm{Tr}^*(m^0 \otimes a^0_\lambda, ..., m^n \otimes a^n_\lambda) = \mathrm{tr}(m^0...m^n)(a^0_\lambda, ..., a^n_\lambda)$, where $(a^0_\lambda, ..., a^n_\lambda) \in \varprojlim \Omega_n(\overline{A}_\lambda)$.

It is easy to see that the homomorphism Tr^* commutes with all the basic operators b^*, b'^*, λ^*, S^*, N^*, B^* and sends reduced entire chains to chains of the same type. Thus, we have a morphism of (B^*, b^*)-bicomplexes of reduced entire chains.

Derivation is a continuous linear map $\delta_\lambda : \overline{A}_\lambda \to \overline{A}_\lambda$ satisfying the Leibniz rule: $\delta_\lambda(a_\lambda b_\lambda) = a_\lambda \delta_\lambda(b_\lambda) + \delta_\lambda(a_\lambda) b_\lambda, \forall \lambda \in I$. We can define a map $L^*_{\delta_\lambda} : C^0_n(A) \to C^0_n(A)$ (or $L^*_{\delta_\lambda} : \varprojlim \Omega_n(\overline{A}_\lambda) \to \varprojlim \Omega_n(\overline{A}_\lambda)$) by the formula

$$L^*_{\delta_\lambda}(a^0_\lambda, ..., a^n_\lambda) = \sum_{j=0}^{n} (a^0_\lambda, ..., \delta_\lambda a^j_\lambda, ..., a^n_\lambda).$$

The map $L^*_{\delta_\lambda}$ is the so-called Lie derivative, associated to derivation δ_λ. It is easy to see that the map $L^*_{\delta_\lambda}$ is a morphism of the (B^*, b^*)-bicomplexes.

Recall that the inner derivation defined by an element $a_\lambda \in \overline{A}_\lambda$ is defined by the formula $\delta_{a_\lambda}(b_\lambda) = [a_\lambda, b_\lambda] = a_\lambda b_\lambda - b_\lambda a_\lambda, \forall b_\lambda \in \overline{A}_\lambda$.

It is easy to see that $\Theta^* \circ L^*_{a_\lambda} = L^*_{a_\lambda} \circ \Theta^*$, where Θ^* is an isomorphism of chain complexes in [6]. It is not difficult to prove that the inner derivatives act trivially on the entire current cyclic homology of (nonunital) involutive Banach algebra.

The map $\theta : \widetilde{\overline{A}}_\lambda \to \widetilde{\overline{A}}_\lambda$, such that $\theta(x_\lambda) = u_\lambda x_\lambda u_\lambda^{-1}$, for every $x_\lambda \in \widetilde{\overline{A}}_\lambda$, and $u_\lambda \in \widetilde{\overline{A}}_\lambda$, $u^2_\lambda = 1$ is an *inner automorphism.* We can write $u_\lambda = -i \exp \frac{\pi i}{2} u_\lambda$. Consider the family of invertibles

$$u_{\lambda_t} = \exp \frac{\pi i t}{2} u_\lambda = \cos \frac{\pi t}{2}.1 + i \sin \frac{\pi t}{2} u_\lambda,$$

for $0 \leq t \leq 1$, and $\theta_t(x_\lambda) = u_{\lambda_t} x_\lambda u_{\lambda_t}^{-1}$, the associated family of inner automorphism. Note that $\theta_0 = id$ and $\theta_1 = \theta$.

In [4], the homomorphism $\theta^* : C^n(\widetilde{\overline{A}}_\lambda, \tau_\lambda) \to C^n(\widetilde{\overline{A}}_\lambda, \tau_\lambda)$ is a morphism of the bicomplexes. It is easy to check that the homomorphism θ^* can be extended to the corresponding operator also denoted by $\theta^* : \varinjlim C^n(\widetilde{\overline{A}}_\lambda, \tau_\lambda) \to \varinjlim C^n(\widetilde{\overline{A}}_\lambda, \tau_\lambda)$, for all fixed n.

We have the operator $\theta_* : C^0_n(\widetilde{A}) \to C^0_n(\widetilde{A})$, which is just the adjoint of the operator θ^* and is a morphism of bicomplexes of reduced entire chains.

It is easy to see that the morphism $\theta_* : B^0_e(\widetilde{A})_{\mathrm{red}} \to B^0_e(\widetilde{A})_{\mathrm{red}}$ induces the identity map on the entire current cyclic homology.

The map $i_\lambda : \overline{A}_\lambda \to M_2(\overline{A}_\lambda)$ is the *homomorphism* that puts $\overline{A}_\lambda$ in the upper left corner of $M_2(\overline{A}_\lambda)$ for all $\lambda \in I$.

It is easy to check that the operator i^*_λ can be extended to the operator denoted by the same character i^*_λ on $\varinjlim C^n(M_2(\overline{A}_\lambda), \tau_\lambda)$ for all fixed n. Now, we can define continuous homomorphisms

$$i_{\lambda *} : C^0_n(A)_{\mathrm{red}} \to C^0_n(M_2(A))_{\mathrm{red}},$$

which are just the adjoint of i^*_λ, for every fixed n.

Proposition 1. $Tr^* \circ i_{\lambda *} = id$.

One can obtain this equality by direct calculation.

Proposition 2. *Let A be an involutive Banach algebra and* $\{A_\lambda\}_{\lambda \in I}$ *the family of ideals with adA-invariant trace in A. The inner automorphism* $\theta : \overline{A}_\lambda \to \overline{A}_\lambda$, *for all* $\lambda \in I$ *induces the identity map on the entire current cyclic homology.*

The idea of the proof is to consider inner automorphism Θ on $M_2(\widetilde{A_\lambda})$, which is defined by an element $U_\lambda = \begin{pmatrix} u_\lambda & 0 \\ 0 & u_\lambda^{-1} \end{pmatrix}$. It is easy to check that $\mathrm{Tr}^* \circ i_{\lambda *} \circ \theta_* = \mathrm{Tr}^* \circ \Theta_* \circ i_{\lambda *}$, where Θ_* is just the adjoint of Θ^* defined by an element U_λ. We have $\Theta_* = \Theta_{1*} \circ \Theta_{2*}$, where Θ_1 and Θ_2 are inner automorphisms defined by elements U_{λ_1}, U_{λ_2} of square 1.

Theorem 1. (Morita invariance) *Let A be an involutive Banach algebra and* $\{A_\lambda\}_{\lambda \in I}$ *the family of ideals with adA-invariant trace in A. The continuous homomorphism* $i_\lambda : \widetilde{A}_\lambda \to M_q(\widetilde{A_\lambda})$, *which puts* $\widetilde{A}_\lambda$ *in the upper-left corner of* $M_q(\widetilde{A_\lambda})$ *for all* $\lambda \in I$, *induces an isomorphism on entire current cyclic homology.*

Sketch of proof. Let $(f_{2n})_{n \geq 0}$ be an even entire reduced cycle in $B^0(M_q(\widetilde{A}))_{\mathrm{red}}$ and $(\omega_{2n})_{2n \geq 0}, \omega_{2n} = (a_\lambda^0 \otimes m^0, \ldots, a_\lambda^{2n} \otimes m^{2n})$ the corresponding forms (see [6]). It is easy to check that $(i_{\lambda *} \circ \mathrm{Tr}^*)(a_\lambda^0 \otimes m^0, \ldots, a_\lambda^{2n} \otimes m^{2n}) = \mathrm{tr}(m^0 \ldots m^{2n})(a_\lambda^0 \otimes e^{00}, \ldots, a_\lambda^{2n} \otimes e^{00})$ $= (\mathrm{Tr}^* \circ \theta_* \circ i_{\lambda *})\,(a_\lambda^0 \otimes m^0, \ldots, \ldots, a_\lambda^{2n} \otimes m^{2n})$, where e^{00} is the elementary matrix with 1 in position (1, 1) and 0 elsewhere. Hence, we have $i_{\lambda *} \circ \mathrm{Tr}^* = \mathrm{Tr}^* \circ \theta_* \circ i_{\lambda *}$. From Propositions 1 and 2, $i_{\lambda *} : HE_*(A) \to HE_*(M_q(A))$ is an isomorphism with the inverse given by Tr^*.

The results of this note will be published in detail elsewhere.

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