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Short Communication

Computation of the Comass of a k-covector

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1. Introduction

The method for solving optimization problems in geometry and other problems by using differential forms of comass one (calibrations) was adopted first by Dao Trong Thi (cf. [14, 15]) and later proposed by Harvey and Lawson [7]. Corresponding to a calibration is a geometry of minimal surfaces (cf. [5, 7]).

The constant coefficient calibrations have been studied by Harvey, Lawson, Morgan, Dadok, etc. Only few examples of calibrations (especially of high degree) are known. Such well-known calibrations are complex line, special Lagrangian, power of Kahler forms (cf. [3, 7]).

By using the method of decomposition of a k-covector with respect to a vector, we describe $F^*(SLAG)$ on $\mathbf{R}^8$ and construct new calibrations on $\mathbf{R}^{4n-1}$.

2. Decomposition of a k-covector with Respect to a Vector

Let Φ be a k-covector on $\mathbf{R}^n (k < n)$ and suppose $\{e_1, e_2, \dots, e_n\}$ is an orthonormal basis of $\mathbf{R}^n$. Denote

$$\Phi_{e_i} = e_i \lrcorner \Phi \qquad i = 1, 2, \dots, n,$$

and

$$\overline{\Phi} = (\Phi_{e_1}, \Phi_{e_2}, \dots, \Phi_{e_n}).$$

We have

$$\|\Phi\|^* = \|\overline{\Phi}\| = \max_{\eta \in G(k-1, \mathbf{R}^n)} \|\overline{\Phi}(\eta)\|,$$

and if $\|\Phi\|^* = 1$, then $G(\Phi) = \{\overline{\Phi}(\eta) \wedge \eta \ / \ \overline{\Phi}(\eta) = 1\}$.

Suppose Φ is a k-covector on $\mathbf{R}^n$ with $\text{span}(\Phi)^* = \mathbf{R}^n$ ($\text{span}(\Phi)^* = \{v \in$ $\mathbf{R}^n / v \perp \Phi = 0\}^{\perp}$) and let e be a unit vector on $\mathbf{R}^n$.

We have the following decomposition of Φ with respect to e:

$$\Phi = e^* \wedge \varphi + \psi,$$

vhere φ and ψ are (k-1) - covector and k - covector on $e^{\perp}$, respectively.

The following lemma gives a relationship between $\|\Phi\|^*$, $\|\varphi\|^*$ and $\|\psi\|^*$.

.emma 2.1.

$$\max\{\|\varphi\|^*, \|\psi\|^*\} \leq \|\Phi\|^* \leq \sqrt{\|\varphi\|^{*2} + \|\psi\|^{*2}}.$$

More exactly, we have the following theorem.

Theorem 2.2.

1) $\|\Phi\|^* = \max\limits_{\eta \in G(k-1, e^{\perp})} \{\sqrt{\varphi(\eta)^2 + \overline{\psi}(\eta)^2}\} = A.$

2) $G(\Phi) = \{(\cos\alpha e + \sin\alpha f) \wedge \eta\}$, *where*

(i) $\varphi(\eta)^2 + \overline{\psi}(\eta)^2 = A^2$;

(ii) $f = \frac{\overline{\psi}(\eta)}{\|\overline{\psi}(\eta)\|}$;

(iii) $\cos\alpha = \frac{\varphi(\eta)}{A}$, $\sin\alpha = \frac{\|\overline{\psi}(\eta)\|}{A}$.

The following corollary is deduced directly from the proof of Theorem 2.2.

orollary 2.3. *Suppose Φ has the following decomposition with respect to e:*

$$\Phi = e^* \wedge \varphi + \psi,$$

here $\|\varphi\|^* = \|\psi\|^* = 1$. Then we have $\|\Phi\|^* = 1$ if and only if $\sqrt{\varphi(\eta)^2 + \overline{\psi}(\eta)^2} \leq 1$ r all $\eta \in G(k-1, \mathbf{R}^n)$.

Application

1. $F^*(SLAG)$ on $\mathbf{R}^8$

enote by $F^*(SLAG)$ the set of all calibrations on $\mathbf{R}^n$, whose faces contain a special agrangian face. The first Cousin principle shows that such calibrations must be of the rm

$$\Phi = \Phi_{SLAG} + \lambda({e^*}_{14} + {e^*}_{25} + {e^*}_{36}) \wedge {e^*}_7 + a.{e^*}_1 \wedge {e^*}_{78}\,.$$

y using the decomposition of Φ with respect to a vector e_8, a direct computation shows at $\Phi \in F^*(SLAG)$ if and only if $a^2 + \lambda^2 \leq 1$.

2. Classification of $F^*(SLAG)$ on $\mathbf{R}^8$

ach $\Phi \in F^*(SLAG)$ is one of the four following types:

(1) Assoc – calibration ($\lambda = \pm 1, a = 0$);
(2) $\Phi = \Phi_{SLAG} + e^*{}_{178}$ ($\lambda = 0, a = \pm 1$). In this case, $G(\Phi) = G(\Phi_{SLAG}) \bigcup CP^2$;
(3) If $\lambda^2 + a^2 < 1$, then $G(\Phi) = G(\Phi_{SLAG})$;
(4) If $\lambda^2 + a^2 = 1$, ($\lambda \neq 0$ and $a \neq 0$), then $G(\Phi) = G(\Phi_{SLAG}) \bigcup B$. B is the set of all 3-vector of the form $(\cos\alpha e + \sin\alpha f) \wedge \eta$, where $f = \frac{\overline{\psi}(\eta)}{\|\psi(\eta)\|}$, and each η is of the form $e_1 \wedge (a_2 e_2 + a_3 e_3 + a_5 e_5 + a_6 e_6 + a_7 e_7)$ in which $a_7 \neq 0$.

4. General Associative Calibrations and General Coassociative Calibrations

Let $\{e_1, Je_1, e_2, Je_2, \ldots, e_{2n}, Je_{2n}\}$ denote the orthonormal basis on $\mathbf{C}^{2n}$ corresponding to the complex structure J.

4.1. General Associative Calibrations

General associative calibrations are calibrations of degree $(2n-1)$ on $\mathbf{R}^{4n-1}$, and associative calibration is a special case when $n = 2$.

Let $\xi \in G(2n-1, Je_n \oplus \mathbf{C}^{2n-1})$. Suppose ξ has the canonical form with respect to the subspace span(Je_n)

$$\xi = (\cos\alpha Je_n + \sin\alpha f) \wedge \eta,$$

where $f \in \mathbf{C}^{2n-1}$, $\eta \in G(2n-1, Je_n \oplus \mathbf{C}^{2n-1})$.

Definition 4.1. *ξ is called G-associative if the following equality holds:*

$$\sum_{k=1}^{p-1} \sum_{|I|=2k}{}' |dZ^I \wedge \Omega_{p-k}(\eta)|^2 = 0.$$

By using the lemma on strengthening of the Wirtinger inequality (see [7]), we have

Theorem 4.2. *The (2n-1)-covector on span$(Je_n) \oplus \mathbf{C}^{2n-1} \simeq \mathbf{R}^{4n-1}$*

$$\Phi_{\text{G-assoc}} = Je_n^* \wedge \Omega_{n-1} + Re\, dZ$$

has comass one, i.e.,

$$\Phi_{\text{G-assoc}}(\xi) \leq |\xi| \text{ for all } \xi \in G(2n-1, \mathbf{R}^{4n-1}),$$

and the equality holds iff ξ is G-associative.

4.2. General Coassociative Calibrations

General coassociative calibrations are calibrations of degree $2n$ on $\mathbf{R}^{4n-1}$, and coassociative calibration is a special case when $n = 2$.

Let $\xi \in G(2n, Je_n \oplus \mathbf{C}^{2n-1})$. Suppose ξ has the canonical form with respect to the subspace span(Je_n)

$$\xi = (\cos\alpha Je_n + \sin\alpha f) \wedge \eta,$$

where $f \in \mathbf{C}^{2n-1}$, $\eta \in G(2n-1, Je_n \oplus \mathbf{C}^{2n-1})$.

Definition 4.3. *ξ is called G-coassociative if the following equalities hold*
(1) $|\mathrm{Re}\, dZ(\eta)|^2 = 0$;
(2) $\sum_{k=1}^{2(n-1)} \sum_{I=2k+1} |dZ^I \wedge \Omega_{2k+1}(\eta)|^2 = 0$.

Also, by using the lemma on strengthening of the Wirtinger inequality (see [7]), we have

Theorem 4.4. *The 2n-covector on span$(Je_n) \oplus \mathbf{C}^{2n-1} \simeq \mathbf{R}^{4n-1}$*

$$\Phi_{\text{G-coassoc}} = Je_n^* \wedge Im\, dZ + \Omega_n$$

has comass one, i.e. ,

$$\Phi_{\text{G-coassoc}}(\xi) \leq |\xi| \;\; for\ all\ \xi \in G(2n, \mathbf{R}^{4n-1}),$$

and the equality holds iff ξ is G-coassociative.

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