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Survey

Generalized Convexity and Some Applications to Vector Optimization*

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Abstract. In this paper we present an overview of recent developments on the characterizations of convex and generalized convex functions via the nonsmooth analysis approach. Generalized convex vector functions are also considered together with their applications to vector optimization.

1. Introduction

Convex sets and convex functions have been studied for some time by Hölder [19], Jensen [20], Minkowski [39] and many others. Due to the works of Fenchel, Moreau, Rockafellar in the 1960s and 1970s, convex analysis became one of the most beautiful and most developed branches of mathematics. It has a wide range of applications including optimization, operations research, economics, engineering, etc. However, several practical models involve functions which are not exactly convex, but share certain properties of convex functions. These functions are a modification or generalization of convex functions. The first generalization is probably due to de Finetti (1949) who introduced the notion of quasiconvexity. Other types of generalized convex functions were later developed in the works of Tuy [45], Hanson [17], Mangasarian [37], Ponstein [41], Karamardian [21], Ortega–Rheinboldt [40], Avriel–Diewert–Schaible–Zang [2].

This paper is an overview of recent developments on the study of convex and generalized convex functions. Our emphasis is on characterizations of these functions by the nonsmooth analysis approach, namely, by subdifferential and directional derivatives. The second part of this overview is to study vector functions, mainly convex vector functions. We shall see that presently, the study of generalized convex vector functions is quite far from being satisfactory. There still remain open problems in relation to the

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definitions of generalized convex vector functions, their structure and characterizations. Nevertheless, many results have already been obtained in the applications of generalized convex vector functions to multi-objective optimization. Because of the survey character of the paper, proofs are not provided except for those of unpublished results.

2. Part A: Generalized Convex Functions

2.1. Most Important Generalized Convex Functions

Let A be a subset of a finite-dimensional space R^n. It is said to be convex if it contains the whole interval linking any two points of A, that is, $\lambda a + (1-\lambda)b \in A$ whenever $a, b \in A$ and $\lambda \in (0,1)$. In this section, f is assumed to be a real function defined on a nonempty convex set $A \subseteq R^n$.

2.1.1. Convex Functions

Definition 1. *The function f is said to be convex if, for every $x, y \in A$ and $\lambda \in (0,1)$, one has*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y). \tag{1}$$

If inequality (1) holds strictly for every $x \neq y$, we say that f is strictly convex on A.

Convex functions are a direct generalization of affine functions because, for the latter, (1) becomes an equality. Convex functions possess many interesting properties. The reader is referred to Rockafellar [42] for a complete study of these functions. Let us recall some properties that will be needed in understanding generalized convex functions.

Remember that the epigraph of f is the set

$$\operatorname{epi} f := \{(x,\alpha) \in R^n \times R : x \in A,\ f(x) \leq \alpha\}$$

and the level set at $\alpha \in R$ is the set

$$\operatorname{lev}(f;\alpha) := \{x \in A : f(x) \leq \alpha\}.$$

Proposition 2. *The function f is convex if and only if one of the following conditions holds:*

(i) $\operatorname{epi} f$ *is convex;*

(ii) $\operatorname{lev}(f+\xi;\alpha)$ *is convex for all $\alpha \in R$ and all $\xi \in R^n$ where the sum function $f+\xi$ is defined by $(f+\xi)(x) = f(x) + \langle \xi, x\rangle$.*

It is worth noting that according to Proposition 2(ii), convex functions have convex level sets; however, the converse is not true. A generalized convexity will be defined to meet this converse. The next result of Fenchel [13] and Mangasarian [38] deals with differentiable functions.

Proposition 3. *Assume A is open and f is differentiable on A. Then f is convex (resp. strictly convex) if and only if, for each $x, y \in A$, $x \neq y$, one has*

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0$$
$$(\text{resp. } \langle \nabla f(x) - \nabla f(y), x - y \rangle > 0).$$

For nondifferentiable functions, one uses the subdifferential to characterize convexity. We recall that the subdifferential of a function f at $x \in A$ is defined by

$$\partial f(x) := \{\xi \in R^n : f(y) - f(x) \geq \langle \xi, y - x \rangle \quad \text{for all } y \in A\}.$$

For convex functions, the subdifferential is nonvoid at every relative interior point of A.

Proposition 4. *Assume $\partial f(x)$ is nonempty for $x \in A$. Then f is convex (resp. strictly convex) if and only if for each $x, y \in A$, $x \neq y$ and $\xi \in \partial f(x)$, $\eta \in \partial f(y)$, one has*

$$\langle \xi - \eta, x - y \rangle \geq 0,$$
$$(\textit{resp. } \langle \xi - \eta, x - y \rangle > 0).$$

The above result is due to Rockafellar [42]. It has been recently extended by Correa–Jofré–Thibaudt and by Luc (see [29]) for the case where f is defined on a Banach space and ∂f is defined in a more general form, including Rockafellar–Clarke's subdifferential, Michel–Penot's subdifferential, etc.

Now, we turn to a subclass of strictly convex functions which is frequently used in optimization and economics.

Definition 5. *f is said to be strongly convex if there exists $\alpha > 0$ such that*

$$f(\lambda x(1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y) - \lambda(1 - \lambda)\alpha \|x - y\|^2$$

for every $x, y \in A$, $\lambda \in (0, 1)$.

Equivalently, f is strongly convex if there is $\alpha > 0$ such that the difference function $f(x) - \alpha\|x\|^2$ is convex.

Using Proposition 4, we obtain the following characterization of strong convexity.

Proposition 6. *Assume $\partial f(x)$ is nonempty for $x \in A$. Then f is strongly convex if and only if there exists $\alpha > 0$ such that, for each $x, y \in A$, $x \neq y$ and $\xi \in \partial f(x)$, $\eta \in \partial f(y)$, one has*

$$\langle \xi - \eta, x - y \rangle - \alpha \|x - y\|^2 \geq 0.$$

2.1.2. *Quasiconvex Functions*

Definition 7. *f is said to be quasiconvex if, for every $x, y \in A$, $x \neq y$ and $\lambda \in (0, 1)$, one has*

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}. \tag{2}$$

If (2) holds strictly, then f is said to be strictly quasiconvex. If (2) holds strictly for those x, y with $f(x) \neq f(y)$, then f is said to be semistrictly quasiconvex.

It is evident that strict quasiconvexity implies semistrict quasiconvexity and quasiconvexity. Semistrict quasiconvexity implies quasiconvexity provided the function is lower semicontinuous.

Remember that the level surface of f at $\alpha \in R$ is the set $\mathrm{surf}(f;\alpha) := \{x \in A : f(x) = \alpha\}$.

Proposition 8. *We have the following:*

(i) *f is quasiconvex if and only if $\mathrm{lev}(f;\alpha)$ is convex for all $\alpha \in R$;*

(ii) *Assume f is continuous and A is strictly convex. Then f is semistrictly quasiconvex if and only if $\mathrm{lev}(f;\alpha)$ is convex and either $\mathrm{lev}(f;\alpha) = \mathrm{surf}(f;\alpha)$ or $\mathrm{surf}(f;\alpha)$ is in the boundary of $\mathrm{lev}(f;\alpha)$;*

(iii) *Assume f is continuous. Then f is strictly quasiconvex if and only if $\mathrm{lev}(f;\alpha)$ is strictly convex and $\mathrm{surf}(f;\alpha)$ consists of extreme points of $\mathrm{lev}(f;\alpha)$ only.*

Proof of (iii). It is already known [2] that $\mathrm{lev}(f;\alpha)$ is strictly convex if f is continuous, strictly quasiconvex. If $\mathrm{surf}(f;\alpha)$ contains a non-extreme point of $\mathrm{lev}(f;\alpha)$, then there are $x, y, z \in \mathrm{lev}(f;\alpha)$ such that $\alpha = f(z) \geq \max\{f(x), f(y)\}$, $z \in (x, y)$. On the other hand, by strict quasiconvexity, $f(z) < \max\{f(x), f(y)\}$, a contradiction. Conversely, if $\mathrm{lev}(f;\alpha)$ is strictly convex, $\alpha \in R$, then it follows in particular that f is quasiconvex. Now, if it is not strictly quasiconvex, we can find $x, y \in A$ and $z \in (x, y)$ such that $f(z) = \max\{f(x), f(y)\}$, say, equal to $f(x)$. Then by quasiconvexity, $f(z) = f(u) = f(x)$ for all $x \in [z, x]$. This means that $\mathrm{surf}(f; f(x))$ does not only consist of extreme points of $\mathrm{lev}(f; f(x))$. The proof is complete. ■

Next is a characterization of quasiconvexity in the differentiable case.

Proposition 9. [1] *Assume f is differentiable on an open convex set $A \subseteq R^n$. Then f is quasiconvex if and only if, for every $x, y \in A$, $f(x) \leq f(y)$ implies $\nabla f(y)(x - y) \leq 0$.*

2.1.3. Pseudoconvex Functions

We use Ortega–Rheinbold's definitions where no differentiability assumption is made.

Definition 10. *f is said to be pseudoconvex if, for every $x, y \in A$ and $f(x) < f(y)$, there exists $\beta > 0$ such that*

$$f(\lambda x + (1-\lambda)y) \leq f(y) - \lambda(1-\lambda)\beta \tag{3}$$

for all $\lambda \in (0, 1)$.

f is said to be strictly pseudoconvex if, for every $x, y \in A$, $x \neq y$ and $f(x) \leq f(y)$, there exists β as above.

Observe that convexity implies pseudoconvexity, which implies semistrict quasiconvexity, and strict convexity implies strict pseudoconvexity.

Proposition 11. *For differentiable functions, the above definition is equivalent to the classical one of Mangasarian [37]:*

f is pseudoconvex if, for $x, y \in A$, $f(x) < f(y)$ implies $\nabla f(y)(x - y) < 0$, and f is strict pseudoconvex if, for $x, y \in A$, $x \neq y$, $f(x) \leq f(y)$ implies $\nabla f(y)(x - y) < 0$.

For pseudoconvex functions, no known criteria exist using epigraph or level sets.

2.2. Extrema of Generalized Convex Functions

For convex functions, it is known [13] that any local minimum is global. This property is very important in optimization since most existing theoretical and computational methods allow us to find local minima only. The concept of quasiconvexity is developed under the influence of this property, namely, we have the following proposition.

Proposition 12. *Let f be a real function on $A \subseteq R^n$. Then*
(i) *If f is quasiconvex, then every strict local minimum is strictly global minimum;*
(ii) *If f is continuous, then f is semistrictly quasiconvex if and only if every local minimum is global minimum;*
(iii) *If f is strictly quasiconvex, then it cannot have a local maximum and it may have a unique minimum.*

Actually, we can characterize quasiconvexity using extrema properties of the function on line segments.

In the next proposition, "strict maximum" on $[x, y]$ means that the point is maximum on $[x, y]$ and unique in some neighborhood on $[x, y]$.

Proposition 13. *Let f be a real function on a convex set $A \subset R^n$. Then the following assertions hold true:*
(i) *f is quasiconvex if and only if its restriction on any closed segment attains its maximum at a segment end;*
(ii) *Assume f is lower semicontinuous on A. f is semistrictly quasiconvex if and only if its restriction on any closed segment attains a strict maximum at a segment end whenever it is not constant at that point.*
(iii) *f is strictly quasiconvex if and only if its restriction on any closed segment attains a strict maximum at a segment end.*

Proof. The first and third assertions are directly obtained by using definitions. We show the second assertion. Assume f is semistrictly quasiconvex. Let $[x, y]$ be any segment in A with $x \neq y$. Assume further that f is not constant on this segment, that is, $f(c) \neq f(y)$ for some $c \in [x, y]$. If $f(x) \neq f(y)$, say $f(x) < f(y)$, then by definition, $f(\lambda x + (1-\lambda)y) < f(y)$ for $\lambda \in (0, 1)$ which shows that y is a strict maximum of f on $[x, y]$. Because f is lower semicontinuous and semistrictly quasiconvex, f is quasiconvex. Hence, if $f(x) = f(y)$, one has $f(c) < f(y)$. Again, by definition,

$$f(\lambda x + (1-\lambda)c) < f(x),$$
$$f(\lambda y + (1-\lambda)c) < f(y)$$

for all $\lambda \in (0, 1)$. Hence, x and y are strict maxima of f on $[x, y]$.

Conversely, if f is not semistrictly quasiconvex, then there can be found $x, y \in A$ and $\lambda \in (0, 1)$ such that $f(x) < f(y) \leq f(\lambda x + (1-\lambda)y)$. It follows in particular that f is not constant on $[x, y]$. By the first part, we may assume f is quasiconvex. Hence, f is constant on $[\lambda x + (1-\lambda)y]$ and x, y cannot be strict maxima of f on $[x, y]$. The proof is complete. ■

Another important property of convex functions is that a point $x \in A$ is minimum if and only if $\nabla f(x) = 0$ when f is differentiable, or $0 \in \partial f(x)$ in a more general case. The concept of pseudoconvexity refers to this property.

Proposition 14. [11, 37] *Let f be differentiable on the open convex set $A \subseteq R^n$. If f is (strictly) pseudoconvex and $\nabla f(x) = 0$, then x is a (strict) global minimum of f on A. Moreover, if f is continuously differentiable on A, then f is (strictly) pseudoconvex if and only if $\nabla f(x) = 0$ implies x is a (strict) local minimum of f.*

2.3. Nonsmooth Generalized Convex Functions

2.3.1. Subdifferential Operator and Support Operator

Denote by $\mathcal{S}(m, n)$ the set of set-valued maps from R^m to R^n and by $\mathcal{B}(m \times n)$ the set of single-valued functions from $R^m \times R^n$ to the extended real line $R \cup \{\pm\infty\}$, which are positively homogeneous in the second variable.

Let us define an operator denoted by σ and called support operator from $\mathcal{S}(m, n)$ to $\mathcal{B}(m \times n)$ and another operator denoted by ∂ and called subdifferential operator from $\mathcal{B}(m \times n)$ to $\mathcal{S}(m, n)$ as follows: For $M \in \mathcal{S}(m, n)$, $x \in R^m$, $v \in R^n$

$$\sigma M(x, v) = \sup_{y \in M(x)} \langle v, y \rangle$$

and for $h \in \mathcal{B}(m \times n)$, $x \in R^m$

$$\partial h(x) = \{v \in R^n : h(x, y) \geq \langle v, y \rangle \text{ for all } y \in R^n\}.$$

It can be seen that

$$M(x) \subseteq \partial(\sigma M)(x),$$

in addition, equality holds provided $M(x)$ is closed convex and

$$h(x, y) \geq \sigma(\partial h)(x, y),$$

in additon, equality holds if h is sublinear lower semicontinuous in y.

Examples.

• Let f be a differentiable function on R^n. Then the directional derivative of f can be seen as a function of two variables

$$f'(x, u) = \langle \nabla f(x), u \rangle.$$

Then $\partial f'(x) = \{v : \langle u, v \rangle \leq \langle \nabla f(x), u \rangle$ for all $u \in R^n\}$

$$= \{\nabla f(x)\},$$

that is, the subdifferential of the directional derivative $f'(x, u)$ at x only consists of the gradient $\nabla f(x)$. On the other hand, $x \mapsto \{\nabla f(x)\}$ is an element of $\mathcal{S}(n, n)$.

$$\begin{aligned} \sigma(\nabla f)(x, u) &= \sup_{y \in \{\nabla f(x)\}} \langle v, y \rangle \\ &= \langle \nabla f(x), u \rangle \end{aligned}$$

that is, the support of ∇f is the directional derivative of f.

• Let f be a convex function on R^n. The directional derivative is defined by

$$f'(x; u) = \lim_{t \downarrow 0} \frac{f(x+tu) - f(x)}{t}.$$

Then $\partial f'$ is exactly convex analysis subdifferential of $\partial_c f$:

$$\partial_c f(x) := \{v \in R^n : \langle v, y - x \rangle \leq f(y) - f(x) \text{ for all } y \in R^n\}.$$

• Let f be a locally Lipschitz function on R^n. The Clarke directional derivative is defined by

$$f^0(x; u) = \lim_{\substack{t \downarrow 0 \\ y \to x}} \frac{f(y+tu) - f(x)}{t}.$$

Then the subdifferential ∂f^0 is exactly Clarke's subdifferential (denoted by $\partial_{cl} f$) of f.

2.3.2. *Generalized Monotonicity*

Throughout this subsection, F is a set-valued map from R^n to R^n. We say that F is

(i) monotone (resp. strictly monotone) if, for $x, y \in \operatorname{Dom} F$, $x \neq y$, $x^* \in F(x)$, $y^* \in F(y)$, one has

$$\langle x^*, y - x \rangle + \langle y^*, x - y \rangle \leq 0; \tag{4}$$

(i′) (resp. $\langle x^*, y - x \rangle + \langle y^*, x - y \rangle < 0$);

(ii) quasimonotone if, for $x, y \in \operatorname{Dom} F$, x^*, y^* as above,

$$\min\{\langle x^*, y - x \rangle , \langle y^*, x - y \rangle\} \leq 0; \tag{5}$$

(iii) semistrictly quasimonotone if it is quasimonotone, and if $\langle x^*, y - x \rangle > 0$ and $\varepsilon > 0$, there can be found $z \in (y - \varepsilon(y - x), y)$, $z^* \in F(z)$ such that $\langle z^*, y - x \rangle \neq 0$;

(iv) strictly quasimonotone if it is quasimonotone, and if $x \neq y$ in $\operatorname{Dom} F$, there are $z \in (x, y)$, $z^* \in F(z)$ such that $\langle z^*, y - x \rangle \neq 0$;

(v) pseudomonotone if (5) is strict whenever $\langle x^*, y - x \rangle \neq 0$;

(vi) strictly pseudomonotone if (5) is strict whenever $x \neq y$.

Observe that (i′) $\Rightarrow$ (vi) $\Rightarrow$ (v) $\Rightarrow$ (ii); (vi) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii); (i) $\Rightarrow$ (ii). The inverse implications are generally not true.

The concept of monotone set-valued maps is known in convex and functional analysis. Quasimonotonicity was introduced in [31]. Under a more complicated form, it was given in [12] for Clarke's subdifferential.

Now, we turn to the concept of generalized monotonicity for bifunctions. Let $f \in \mathcal{B}(n, n)$. We say that f is

(i) monotone if, for every $x, y \in R^n$, one has

$$f(x, y - x) + f(y, x - y) \leq 0; \tag{6}$$

(i′) strictly monotone if (6) is strict whenever $x \neq y$;

(ii) quasimonotone if, for every $x, y \in R^n$, one has

$$\min\{f(x, y-x), f(y, x-y)\} \leq 0; \tag{7}$$

(iii) semistrictly quasimonotone if it is quasimonotone, and if $f(x, y-x) > 0$, $f(y, x-y) > -\infty$, $\varepsilon > 0$ implies the existence of $z \in (y-\varepsilon(y-x), y)$ such that $f(z, y-x) > 0$;

(iv) strictly quasimonotone if it is quasimonotone, and if $x, y \in X$, $x \neq y$ with $f(x, y-x) > -\infty$ and $f(y, x-y) > -\infty$ implies the existence of $z \in (x, y)$ such that either $f(z, y-x) > 0$ or $f(z, x-y) > 0$;

(v) pseudomonotone if the inequality in (7) is strict whenever $f(x, y-x) \neq 0$;

(vi) strictly pseudomonotone if the inequality in (7) is strict whenever $x \neq y$.

As in the case of set-valued maps, we have the following implications (i′) ⇒ (vi) ⇒ (v) ⇒ (ii); (vi) ⇒ (iv) ⇒ (iii) ⇒ (ii); (i) ⇒ (ii).

Monotonicity of bifunctions was used in [35]. Generalized monotone bifunctions were developed in [27, 31].

Below we present a relation between generalized monotonicity of a set-valued map and that of its support bifunction.

Theorem 15. *Let $F \in \mathcal{S}(n, n)$. Then*

(i) *F is G-monotone if and only if σF is G-monotone, where G may take the prefix ϕ, quasi, semistrictly quasi, and strictly quasi;*

(ii) *if σF is strictly monotone (resp. pseudomonotone, strictly pseudomonotone), then so is F. The converse is also true if, in addition, F is compact-valued.*

Similarly, a link between generalized monotonicity of a bifunction and its subdifferential is given in the next theorem.

Theorem 16. *Let $f \in \mathcal{B}(n, n)$ be sublinear lower semicontinuous in the second variable. Then*

(i) *f is G-monotone if and only if ∂f is G-monotone, where G may be ϕ, quasi, semistrictly quasi, and strictly quasi;*

(ii) *if f is strictly monotone (resp. pseudomonotone, strictly pseudomonotone), then so is ∂f. The converse is also true if ∂f is compact-valued.*

The merit of the above results is resigned in the fact that in order to establish generalized monotonicity of a set-valued map, one can verify it for its support function or vice versa.

2.3.3. Characterizations of Generalized Convex Functions

In this subsection, we characterize generalized convex functions by means of Clarke–Rockafellar's subdifferential. Other types of subdifferentials can also be used in a similar way.

We recall that Clarke–Rockafellar's subdifferential $\partial^{\uparrow} f$ is defined as in Subsec. 2.3.1 by using the following directional derivative:

$$f^{\uparrow}(x; u) = \sup_{\varepsilon>0} \limsup_{(y,\alpha)\downarrow_f x;\, t\downarrow 0} \inf_{v\in B(u,\varepsilon)} \frac{f(y+tv)-\alpha}{t},$$

where f is a lower semicontinuous function on R^n, $(y, \alpha) \downarrow_f x$ means $y \to x$, $\alpha \to f(x)$ and $\alpha \geq f(y)$; $B(u, \varepsilon) = \{v \in R^n : \|v - u\| < \varepsilon\}$. The above formula takes a simple form when f is locally Lipschitz:

$$f^{\uparrow}(x; u) = \limsup_{y \to x; t \downarrow 0} \frac{f(y + tu) - f(y)}{t}.$$

In Sec. 1 (Proposition 4), we have already formulated a criterion for convex functions. A fundamental result in characterizing generalized convexity via subdifferential is the following theorem of [30].

Theorem 17. *Let f be a lower semicontinuous function from R^n (even a Banach space) to $R \cup \{+\infty\}$. Then f is quasiconvex if and only if $\partial^{\uparrow} f$ is quasimonotone.*

With the aid of this theorem, it is possible to establish subdifferential characterizations of other types of generalized convexity. We shall assume f is locally Lipschitz and dom f is open and convex.

Proposition 18. *The function f is strictly (resp. semistrictly) quasiconvex if and only if the following conditions hold:*
(i) *$\partial^{\uparrow} f$ is quasimonotone;*
(ii) *For $x \neq y$ and $f(x) = f(y)$, there exist $a, b \in (x, y)$ and $a^* \in \partial^{\uparrow} f(x)$, $b^* \in \partial^{\uparrow} f(y)$ such that*

$$\langle a^*, y - x \rangle \cdot \langle b^*, y - x \rangle > 0.$$

(resp.)
(ii′) For $f(x) < f(y)$ and $\varepsilon > 0$, there exist $z \in (y - \varepsilon(y - x), y)$ and $z^* \in \partial^{\uparrow} f(z)$ such that $\langle z^*, y - z \rangle \neq 0$).

In particular, if f is strictly quasiconvex, then $\partial^{\uparrow} f$ is strictly quasimonotone, and if $\partial^{\uparrow} f$ is semistrictly quasimonotone, then f is semistrictly quasiconvex.

Note that the converse of the last assertion in the preceding proposition is also true when $n = 1$, and not true generally in a higher dimension.

Proposition 19. *If the function f is pseudoconvex (resp. strictly pseudoconvex), then*
(i) *$\partial^{\uparrow} f$ is quasimonotone;*
(ii) *for $f(x) < f(y)$ (resp. $f(x) \leq f(y)$ and $x \neq y$), there exists $y^* \in \partial^{\uparrow} f(y)$ such that $\langle y^*, x - y \rangle < 0$.*

Conversely, if $\partial^{\uparrow} f$ is pseudomonotone (resp. strictly pseudomonotone), then f is pseudoconvex (resp. strictly pseudoconvex).

Observe that conditions for generalized convexity presented in Propositions 18 and 19 involve both the values of the function and that of the subdifferential. Subdifferential alone is not sufficient or necessary to characterize generalized convexity except for the quasiconvexity. Because generalized convex functions are determined exclusively by their restrictions on line segments, the classical directional derivatives, which only make use of the values of functions on direction, seem to be more appropriate to produce generalized convexity characterizations.

Proposition 20. *Let f be a lower semicontinuous function on an open convex set $A \subseteq R^n$. Then f is G-convex if and only if its directional upper derivative f^+ is a G-monotone bifunction on A, where G may be ϕ, strictly, quasi, semistrictly quasi, strictly quasi, pseudo, and strictly pseudo.*

The directional lower derivative f^- can also be used in the previous proposition. However, for (strict) pseudoconvexity, the sufficient part may fail.

Let us specify the above result for the case of differentiable functions. Remember that in this case,

$$f^+(x, v) = \langle \nabla f(x), v \rangle.$$

The following conditions are necessary and sufficient for f to be respectively quasiconvex, semistrictly quasiconvex, strictly quasiconvex, pseudoconvex and strictly pseudoconvex. For every $x, y \in A$, $x \neq y$,

(i) $\min\{\langle \nabla f(x), y - x \rangle \,,\, \langle \nabla f(y), x - y \rangle\} \leq 0;$ (8)

(ii) $\langle \nabla f(x), y - x \rangle > 0$ implies $\langle \nabla f(y), x - y \rangle \leq 0$, and in any neighborhood of y on $[x, y]$, there is z such that $\langle \nabla f(z), y - x \rangle > 0$;

(iii) inequality (8) holds and there is $z \in (x, y)$ with $\langle \nabla f(z), y - x \rangle \neq 0$;

(iv) inequality (8) is strict if one of two terms under min is nonzero;

(v) inequality (8) is strict.

The result of (iv) is shown by [21], the results of (i) and (v) were given by [16, 22, 44].

3. Vector Functions and Vector Optimization

3.1. Efficient Points

Let C be a convex closed pointed cone in R^n. It produces a partial order in R^n by

$$a \geq b \quad \text{if} \quad a - b \in C.$$

Let A be a nonempty subset of R^n. A point $a \in A$ is said to be efficient (or minimal with respect to C) if there is no $b \in A$ such that $a \geq b$, $a \neq b$. The set of minimal points of A is denoted by Min A.

When int C is nonempty, one is also interested in the set of weakly efficient points:

$$W\text{Min}\, A := \{a \in A \,:\, \text{there is no } b \in A \text{ with } a - b \in \text{int}\, C\}.$$

Similarly, one defines the set of maximal points and the set of weakly maximal points Max A, WMin A. A particular case is when C is the positive orthant R^n_+ of R^n, i.e., $R^n_+ = \{a = (a_1, ..., a_n) \in R^n : a_i \geq 0,\ i = 1, ..., n\}$. Efficient points with respect to R^n_+ are called Pareto-minimal points.

Occasionally, one also considers ideal points. Recall that $a \in A$ is said to be ideal if $a \leq b$ for all $b \in A$. Generally, ideal points exist under very restrictive conditions.

3.2. Convex Vector Functions

Let X be a convex subset of R^m and f a function from X to R^n. As in the previous section, R^n is ordered by a convex closed pointed cone C.

We say that f is convex if, for $x,\ y \in X,\ \lambda \in (0,1)$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

In the case int $C \neq \emptyset$, by writing $a \gg b$, we mean $a - b \in \text{int}\, C$. Then f is strictly convex if, for $x \neq y$, λ as above, one has

$$f(\lambda x + (1-\lambda)y) \ll \lambda f(x) + (1-\lambda) f(y).$$

A useful property of convex functions is given in the next lemma.

Lemma 21. *f is convex (resp. strictly convex) if and only if $\xi \circ f$ is convex (resp. strictly convex) for every $\xi \in C' \setminus \{0\}$, where C' is the dual cone of C, i.e., $C' = \{\xi \in R^n : \langle \xi, x\rangle \geq 0,\ x \in C\}$.*

Let us define subdifferential for a convex vector function in a standard way.

$$\partial f(x) := \{A \in L(m,n) : f(y) - f(x) \geq A(y-x) \text{ for all } y \in X\},$$

where $L(m,n)$ denote the space of $m \times n$-matrices. One can prove several properties of subdifferential of convex vector functions similar to those of scalar convex functions. For instance, $\partial f(x)$ is a convex closed set. If f is differentiable, then $\{\nabla f(x)\} = \partial f(x)$.

Moreover, using Lemma 21, one can establish the following relation: $\partial(\xi \circ f)(x) \supseteq \xi \partial f(x)$ for every $\xi \in C'$ and equality holds when $x \in ri\, X$ (the relative interior of X).

Now, assume f is locally Lipschitz. We recall the generalized Jacobian

$$Jf(x) := \overline{\text{co}}\{\lim \nabla f(x_i) : x_i \in X,\ \ f \text{ is differentiable at } x_i \text{ and } x_i \to x\},$$

where $\overline{\text{co}}$ denotes the closed convex hull.

It is known that if f is scalar-valued, then $Jf(x)$ is the Clarke subdifferential and coincides with the convex analysis subdifferential $\partial f(x)$. This fact is no longer true for vector functions. Nevertheless, we have the following relation established by [36].

Proposition 22. *Let $x \in \text{int}\, X$. Then $Jf(x) \subseteq \partial f(x)$ and $\xi Jf(x) = \xi \partial f(x)$ for all $\xi \in C'$.*

The equality is obtained by the fact that $J(\xi \circ f)(x) = \xi Jf(x)$ and $\partial(\xi \circ f) = \xi \partial f$. The composition $\xi \circ f$ is a convex scalar function when $\xi \in C'$. The inclusion in Proposition 22 is strict. For instance, with $f(x,y) = (|x|, |x| + y)$ from R^2 to R^2 and $C = R^2_+$, one has $\partial f(0,0) = \{\left(\begin{smallmatrix}\alpha & 0\\ \beta & 1\end{smallmatrix}\right),\ \alpha, \beta \in [0,1]\}$, while $Jf(0,0) = \overline{\text{co}}\{\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1 & 0\\ -1 & 1\end{smallmatrix}\right)\}$.

To study convex vector functions, we also need the concept of monotone maps. Let F be a set-valued map from $X \subseteq R^m$ to $L(m,n)$. Remember that R^n is ordered by the cone C. We say that F is monotone if, for $x, y \in X,\ \ A \in F(x),\ \ B \in F(y)$, one has $(A - B)(x - y) \geq 0$.

It is easy to see that F is monotone if and only if $\xi \circ F$ is monotone in the classical sense for every $\xi \in C'$.

Now, let f be a vector function, not necessarily convex. One defines subdifferential of f as if f were convex, that is,

$$\partial f(x) := \{A \in L(m,n) : f(y) - f(x) \geq A(y-x) \text{ for all } y \in X\}.$$

The following result was partly proved in [36].

Proposition 23. *Let f be a vector function from $X \subseteq R^m$ to R^n. If $\partial f(x)$ is nonvoid for every $x \in X$, then f is convex. Moreover, if f is convex, then ∂f is maximal monotone.*

Proof. If $\partial f(x)$ is nonvoid for all $x \in X$, then given $x, y \in X$, $\lambda \in (0,1)$, one has

$$f(x) - f(\lambda x + (1-\lambda)y) \geq (1-\lambda)A(x-y),$$
$$f(y) - f(\lambda x + (1-\lambda)y) \geq \lambda A(y-x),$$

where A is any element of $\partial f(\lambda x + (1-\lambda)y)$. It follows from the above inequalities that $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$. Hence, f is convex. The second part is proven in a standard way.

For a locally Lipschitz function, the generalized Jacobian is always nonvoid. We can use it to characterize convexity. ■

Proposition 24. *Let f be a locally Lipschitz function from an open convex set $X \subseteq R^m$ to R^n. Then f is convex if and only if Jf is monotone.*

As a consequence of Propositions 23 and 24, we see that generalized Jacobian of a convex vector function is not maximal monotone in general.

Now, we study optimality conditions of convex functions. Recall that the tangent cone of a convex set $X \subseteq R$ at $x_0 \in X$ is the cone

$$T(X; x_0) := \{v \in R^m : x_0 + tv \in X \text{ for some } t > 0\}.$$

We say that $x_0 \in X$ is an efficient point of f if $f(x_0)$ is an efficient point of the set $f(X)$. Ideal points and weakly efficient points of f are defined in a similar way. For a set $Q \subseteq R^m$, we denote $Q^\# = \{x \in R^m : \langle x, q\rangle \geq 0 \text{ for some } q \in Q\}$.

Proposition 25. *Let f be a convex vector function on a convex set $X \subseteq R^m$. Then*
(i) *$x_0 \in X$ is ideal if and only if $0 \in \partial f(x_0)$;*
(ii) *$x_0 \in X$ is weakly efficient if and only if there is $\xi \in C' \setminus \{0\}$ such that $T(X; x_0) \subseteq [\xi \partial f(x_0)]^\#$.*

Proof. The first assertion is immediate from definitions. Let $x_0 \in X$ be weakly efficient. Then there exists $\xi \in C' \setminus \{0\}$ such that x_0 is a minimum of $\xi \circ f$ over X. Since $\xi \circ f$ is convex, we have, for every $v \in T(X; x_0)$: $\sup_{\eta \in \partial \xi \circ f(x_0)} \langle \eta, v\rangle \geq 0$. Equivalently, $\langle \xi \in A, v\rangle \geq 0$ for some $A \in \partial f(x_0)$. Thus, $T(X; x_0) \subseteq [\xi \partial f(x_0)]^\#$. Conversely, if x_0 is not weakly efficient, then there is $v \in T(X; x_0)$ such that $f(x_0 + v) - f(x_0) \ll 0$. Hence, for every $\xi \in C' \setminus \{0\}$, $\xi \circ f(x_0 + v) - \xi \circ f(x_0) < 0$. Consequently, $\langle \eta, v\rangle \leq \xi \circ f(x_0 + v) - \xi \circ f(x_0) < 0$ for every $\eta \in \partial \xi \circ f(x_0)$, which means that $v \notin [\xi \partial f(x_0)]^\#$. ■

A particular case of Proposition 25 is the situation where x_0 is a relative interior point of X, i.e., $x_0 \in ri X$. Then x_0 is weakly efficient if and only if $[Lin\ X]^\perp \cap \xi \partial f(x_0) \neq \emptyset$ for some $\xi \in C' \setminus \{0\}$, where $[Lin\ X]^\perp$ is the orthogonal complement of the linear subspace spanned on $X - x_0$. For $X = R^m$, the above relation is equivalent to $0 \in \xi \partial f(x_0)$.

Another particular case is obtained when f is differentiable at x_0. We then have $\partial f(x_0) = \{f'(x_0)\}$ and Proposition 25(ii) becomes as follows: $x_0 \in X$ is weakly efficient if and only if there is $\xi \in C' \setminus \{0\}$ such that

$$\langle \xi f'(x_0), v \rangle \geq 0 \text{ for all } v \in T(X; x_0).$$

Again, if in addition x_0 is a relative interior point of X, then the above relation is equivalent to

$$\xi f'(x_0) \in [Lin\ X]^{\perp}$$

and, for $X = R^m$, it means that $0 = \xi f'(x_0)$.

3.3. Quasiconvex Vector Functions

As in the previous section, f is a vector function from a convex set $X \subseteq R^m$ to R^n and the space R^n is partially ordered by a closed convex pointed cone C.

Definition 26. *We say that f is (strict) quasiconvex if, for $x, y \in X$, $x \neq y$ and for $\lambda \in (0, 1)$, one has*

$$f(\lambda x + (1-\lambda)y) \leq [f(x), f(y)]^+$$

$$(\textit{resp. } f(\lambda x + (1-\lambda)y) \ll [f(x), f(y)]^+),$$

where $[f(x), f(y)]^+ := \{a \in R^n : a \geq f(x) \text{ and } a \geq f(y)\}$.

This definition of quasiconvexity has the advantage in that quasiconvex vector functions can completely be characterized by level sets. We recall that, for $a \in R^n$, $\mathrm{lev}(f; a)$ consists of $x \in X$ such that $f(x) \leq a$.

Proposition 27. *f is quasiconvex if and only if level sets of f are convex.*

The definition given above is one way to generalize the relation (2) to vector functions. That relation can also be interpreted in other ways, giving different definitions of quasiconvex vector functions. Let us mention some of them:

(a) $f(x) \leq f(y)$ implies $f(\lambda x + (1-\lambda)y) \leq f(y)$.

(b) $f(\lambda x + (1-\lambda)y) \leq \beta f(x) + (1-\beta) f(y)$ for some $\beta \in (0, 1)$. (Definition by Tanaka and Helbig.)

It is easily seen that f is quasiconvex in the sense of (a) if and only if $\mathrm{lev}(f; f(x))$ is star-shaped with center at x.

Cambini and Martein [6] go even further in generalizing relation (2) by using separately the orders generated by C, $C \setminus \{0\}$, $\mathrm{int}\, C$ in the inequalities in (a). The interested reader is referred to [4] for details of those generalizations.

As noticed before, a vector function is convex if and only if the composition $\xi \circ f$ is convex for every $\xi \in C'$. This property makes it possible to treat a family of scalar functions $\{\xi \circ f : \xi \in C'\}$ instead of the vector function f. This shows particular importance in scalarization of vector problems. A similar result does not hold for quasiconvex functions; composition of a quasiconvex function with a functional $\xi \in C'$ is not necessarily quasiconvex. When C is the nonnegative orthant of the space R^n, we have the following result: $f = (f_1, ..., f_n)$ is (strictly) quasiconvex if and only if $f_1, ..., f_n$ are (strictly) quasiconvex.

Quasiconvexity is used in vector optimization to establish local-global properties of efficient solutions. It is also used to prove topological properties such as connectedness and contractibility of the set of efficient solutions. Let us consider the case $C = R^n_+$. With this order, the space R^n is a Banach lattice. In particular, for every $a, b \in R^n$, one can find a unique element $u \in R^n$ such that $u \geq a,\ u \geq b$ and, for any other v satisfying the above inequalities, one has $v \geq u$. Such an element u is denoted by $\sup\{a, b\}$. Let us introduce the concept of semistrict quasiconvexity (called explicitly quasiconvexity in [34]). We say that f is semistrictly quasiconvex if it is quasiconvex, and if $f(x) \neq f(y)$, one has $f(\lambda x + (1-\lambda)y) < \sup(f(x), f(y))$. Below is the local-global property in vector optimization.

Proposition 28. *Assume f is semistrictly quasiconvex. Then every local efficient solution of f over X is its global efficient solution.*

Note that if the components $f_1, ..., f_n$ of the function f are semistrictly quasiconvex and quasiconvex, then so is f. The converse is not true in general. The componentwise semistrict quasiconvexity can be used to characterize local-global property for weakly efficient solutions.

Proposition 29. *Assume f is componentwise semistrict quasiconvex (and quasiconvex). Then every local weakly efficient solution of f over X is its global weakly efficient solution.*

The local-global property is also satisfied under pseudoconvexity, a notion introduced by [6]. Assume f is directionally differentiable at x and $\text{int}C \neq \emptyset$. f is said to be pseudoconvex if $f(y) < f(x)$ implies $f'(x, y-x) \ll 0$.

Proposition 30. [6] *If f is pseudoconvex, then every local efficient solution of f over X is a global efficient solution.*

It is known and easy to see that if $x \in X$ is a (weakly) efficient solution of f on X, then $f'(x, v) \notin -\text{int}C$ for every $v \in T(X; x)$. It follows directly from the definition of pseudoconvexity that the above condition is also sufficient for efficiency when f is pseudoconvex. Second order efficiency conditions were developed by [4] using generalized convexity.

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