Vietnam Journal of Mathematics **26**:3 (1998) 229–233

Vietnam Journal
of
MATHEMATICS

Characterization of Rings Using Weakly Projective Modules II

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Received October 31, 1995
Revised January 28, 1996

Abstract. In this paper, some characterizations of PP-rings, semisimple rings, semiperfect rings, and semiregular and F-semiperfect modules by weakly projective modules are given. Our results generalize several well-known results by Golan, Oberst, and Schneider [2, 3].

1. Introduction

Recently, many well-known theorems about projective and quasi-projective modules have been generalized by using weaker properties. An interest has grown for those "projective properties" (cf. [4, 5, 6, 8, 9]). In [6], we gave characterization of rings by weakly projective modules and posed the following problem: Does Morita equivalence preserve weakly projective modules? In this paper, we prove that the notion of weakly projective modules is Morita invariant. Using this property we give some new characterizations of rings using weakly projective properties. So this paper can be considered as a continuation of [6].

Throughout, all rings considered have an identity and modules are unital left modules. We will freely make use of the notations, terminologies and results of [1, 2, 6, 7].

Following Zoschinger [9], we call a module M weakly projective if, for every pair (A, B) of submodules of M with $M = A+B$, there exists an endomorphism $f : M \to M$ such that $\mathrm{Im}(f) \subseteq A$ and $\mathrm{Im}(1-f) \subseteq B$. Also, M is called direct-projective [8] if, given any summand N of M with projection $p : M \to N$ and any epimorphism $f : M \to N$, there exists $g \in \mathrm{End}(M)$ such that $fg = p$. By [6], the following hierarchy exists:

$$\text{quasi-projective} \Rightarrow \text{underprojective [5]} \Rightarrow \text{weakly projective}$$

The following example shows that weakly projective modules need not be direct projective.

Example 1.1. The Z-module $Z(p^\infty)$ is not direct projective since the multiplication with p on $Z(p^\infty)$ is a non-splitting epimorphism. However, $Z(p^\infty)$ is a weakly projective module.

The following two results are very useful in this paper.

Lemma 1.2. [6, Lemma 1.1] *Let P be projective and $P \oplus M$ weakly projective. If there is an epimorphism h: $P \to M$, then M is projective.*

Theorem 1.3. *Let $F : R$–Mod $\to S$–Mod define a Morita equivalence between the category R–Mod of left R–modules and the category S–Mod of left S–modules. Then an R–module ${}_RM$ is weakly projective if and only if ${}_SF(M)$ is weakly projective.*

Proof. Let $F = \mathrm{Hom}_R({}_RQ_S, -)$, where ${}_RQ$ is a finitely generated projective generator, and define the equivalence. Let M be a weakly projective module. Then each pair of submodules (A', B') of ${}_SF(M)$ such that $A' + B' = F(M)$ is of the form $(F(A), F(B))$, where A, B are submodules of M. Because ${}_RQ$ is finitely generated projective generator, we can prove that $A + B = M$ and hence, there exists an endomorphism $f : M \to M$ such that $\mathrm{Im}(f) \subseteq A$ and $\mathrm{Im}(1 - f) \subseteq B$ since M is weakly projective. Thus, we obtain an endomorphism $F(f) : F(M) \to F(M)$, and ${}_RQ$ is a finitely generated projective generator. We can prove that $\mathrm{Im}(F(f)) \subseteq F(\mathrm{Im}(f)) \subseteq A'$ and $\mathrm{Im}(1 - F(f)) = \mathrm{Im}(F(1 - f)) \subseteq F(\mathrm{Im}(1 - f)) \subseteq B'$. Hence, ${}_SF(M)$ is weakly projective. ∎

2. Characterizing Rings by Weakly Projective Modules

A ring R is left PP if each principal left ideal is projective. We denote by R_n the ring of $n \times n$ matrices over R. If M is an R-module, then M^n is the product of n copies of M.

Proposition 2.1. *The following are equivalent:*

(1) *R is a left PP-ring;*

(2) *every principal left ideal of R_2 generated by a diagonal matrix is weakly direct-projective.*

Proof. (1) $\Rightarrow$ (2) See [2, Lemma 4.2].

(2) $\Rightarrow$ (1) Let $r \in R$ and let I be the principal left ideal of R_2 generated by the diagonal matrix $\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$. Then I is a weakly projective R-module. Since there is a Morita equivalence between R_2-modules and R-modules via $M \to eM$, where M is an R_2-module and $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R_2$. Now, $eI \cong Rr \oplus R$ as R-modules, so $Rr \oplus R$ is weakly projective. Hence, Rr is projective and thus R is left PP. ∎

Semisimple rings were characterized by Goland [1, 2] using quasi-projective modules and by Tiwary and Pandeya [4] using pseudo-projectives. We can use Theorem 1.3 to generalize some of their results.

Theorem 2.2. *The following are equivalent for a ring R:*

(1) *R is semisimple;*

(2) *for all $n \geq 1$, every cyclic R_n-module is weakly projective;*

(3) *there exists some $n > 1$ such that every cyclic R_n-module is weakly projective.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial.

(3) $\Rightarrow$ (1) Let I be a left ideal of R. To show that R/I is projective, we denote by I_n the left ideal of R_n consisting of all matrices with entries from I. Let $e_{ij} \in R_n$ be the matrix unities. Then $R_n/I_n e_{ij} \cong P \oplus M$ as left R_n-modules, where $M = R_n e_{11}/I_n e_{11}$ and $P = \sum_{i=2}^{n} R_n e_{ij}$. Hence, $P \oplus M$ is weakly projective as R_n-module by (3). Clearly, P is projective and there is an R_n-epimorphism $P \to M$ via $(r_{ij}) \mapsto (r_{ij})e_{21} + I_n e_{11}$. Hence, M is a projective R_n-module. By the fact that there is a Morita equivalence between R_n-modules and R-modules via $M \to e_{11}M$, where M is an R_n-module, since M is a projective R_n-module, $e_{11}M \cong R/I$ is a projective R-module. ■

Golan [2] proved that R is left (semi)-perfect if and only if every (finitely generated) module has a quasi-projective cover. In [6], we introduce the following concept:

Definition. *We call an epimorphism $f : Q \to M$ a weakly projective cover of M if Q is weakly projective and Ker f is small in Q.*

Lemma 2.3. [6, Lemma 3.1] *Let P be a projective module. Assume $P \oplus M$ has a weakly projective cover. If there is an epimorphism $f : P \to M$, then M has a projective cover.*

Theorem 2.4. *The following conditions are equivalent for a ring R:*

(1) *R is left semiperfect;*

(2) *for all natural numbers n, every cyclic left R_n-module has a weakly projective cover, where R_n denotes the ring of all $n \times n$ matrices over R;*

(3) *there exists a natural number $n > 1$ such that every cyclic left R_n-module has a weakly projective cover.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial.

(3) $\Rightarrow$ (1) Let I be a left ideal of R. To show that R/I has a projective cover, we denote by I_n the left ideal of R_n consisting of all matrices with entries from I, then I_n is a left ideal of R_n. Let $e_{ij} \in R_n$ be the matrix unities. Then $R_n/I_n e_{ij} \cong P \oplus M$ as left R_n-modules, where $M = R_n e_{11}/I_n e_{11}$ and $P = \sum_{i=2}^{n} R_n e_{ij}$. Clearly, P is projective and there is an R_n-epimorphism $P \to M$ via $(r_{ij})_{i,j=1}^{n} \mapsto (r_{ij})_{j=1}^{n} e_{21} + I_n e_{11}$. Hence, the cyclic R_n-module $R_n/I_n e_{11} \cong P \oplus M$ has a weakly projective cover. By Lemma 2.3, the R_n-module M has a projective cover $f : Q \to M$. Since $e_{11}R_n e_{11} \cong R$, therefore, $f(e_{11}Q) = e_{11}f(Q) = e_{11}M \cong R/I$ as an R-module. Since Q is a projective R_n-module, then $e_{11}Q$ is a projective R-module. Therefore, the R-module epimorphism $e_{11}Q \to R/I$ induced by f is a projective cover of the cyclic R-module R/I. Thus, R is a left semiperfect ring. ■

3. Characterizing Semiregular and F-Semiperfect Modules by Weakly Projective Covers

An R-module M is called semiregular if every finitely generated (cyclic) submodule of M lies over a projective summand of M. A projective module P is called an F-semiperfect module if P/P_s has a projective cover for each $s \in \mathrm{End}({}_RP)$. The ring R is semiregular ($= F$-semiperfect) if ${}_RR$ is a semiregular module. Analogously to Xue [8], we can prove the following results:

Proposition 3.1. *A projective module ${}_RP$ is semiregular if and only if $P \oplus (P/N)$ has a weakly projective cover for each finitely generated (cyclic) submodule N of P.*

Proposition 3.2. *A projective module ${}_RP$ is F-semiperfect if and only if $P \oplus (P/Ps)$ has a weakly projective cover for each $s \in End({}_RP)$.*

Oberst and Schneider [3, Satz 1.2] proved that R is a semiregular ring if and only if each finitely presented left (right) R-module has a projective cover if and only if, for each $r \in R$, the left (right) R-module $R \oplus (R/Rr)$ (resp. $R \oplus (R/rR)$) has a projective cover. Using these results and Lemma 2.3, we can prove the following result:

Proposition 3.3. *The following conditions are equivalent for a ring R:*

(1) *R is a semiregular ring;*

(2) *each finitely presented left (right) R-module has a weakly projective cover;*

(3) *the left (right) R-module $R \oplus (R/Rr)$ (resp., $R \oplus (R/rR)$) has a weakly projective cover for each $r \in R$.*

We also conclude this paper with the following remark:

Remark. We think that many other classes of rings or modules may be characterized by weakly projective modules.

Acknowledgement. The authors are grateful to Professor Weimin Xue for many good suggestions and providing the reprint of [8].

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