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Mixed Boundary-domain Operator in Approximate Solution of Biharmonic Type Equation

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Abstract. In [2], we constructed an iterative method for solving the boundary value problem (BVP) (1)–(3) under the assumptions (4) and (5). In this paper, we consider the case where condition (5) is not satisfied. With the help of a new operator, which acts on couples of boundary and domain functions, we lead BVP (1)–(3) to a mixed boundary-domain operator equation and construct an iterative method for it.

1. Introduction

In [2], we constructed an iterative method for solving the following BVP:

$$\varepsilon\Delta^2 u - a\Delta u + bu = f(x), \quad x \in \Omega, \tag{1}$$

$$u|_\Gamma = 0, \tag{2}$$

$$\left.\frac{\partial u}{\partial \nu}\right|_\Gamma = 0, \tag{3}$$

where Δ is the Laplace operator, Ω a bounded domain in R^n $(n \geq 2)$, Γ the sufficiently smooth boundary of Ω, ν the outward normal to Γ, and $\varepsilon > 0$ under the assumptions

$$a = \text{const} \geq 0, \quad b = \text{const} \geq 0, \tag{4}$$

$$a^2 - 4b\varepsilon \geq 0. \tag{5}$$

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More precisely, the boundary conditions (2) and (3) may be nonhomogeneous, but it is not essential. The latter condition (5) is very important and cannot be missed for reducing the problem (1)–(3) to a boundary operator equation and for establishing the convergence of an iterative process.

In this paper, we consider the case where the condition (5) is not satisfied. In this case, with the help of a new operator which acts on couples of boundary and domain functions, we lead BVP (1)–(3) to a mixed boundary-domain operator equation and construct an iterative method for it.

2. Reduction of BVP to Boundary Operator Equation

For the sake of simplicity, we limit our discussion to the case $\varepsilon = 1$ because our purpose is to construct an approximate solution of (1)–(3) associated with some fixed ε but not to consider its effect when tending to zero. It is the scope of the theory of singular perturbation problems.

Now, we set

$$\begin{aligned} \Delta u &= v \\ D &= -bu \end{aligned} \tag{6}$$

and denote by v_0 the trace of v on Γ and

$$Lv = \Delta v - av. \tag{7}$$

Then we come to the sequence of problems

$$\begin{aligned} Lv &= f + D, \quad x \in \Omega, \\ v|_\Gamma &= v_0, \end{aligned} \tag{8}$$

$$\begin{aligned} \Delta u &= v, \quad x \in \Omega, \\ u|_\Gamma &= 0. \end{aligned} \tag{9}$$

The solutions u and v of the above problems should satisfy the relations (3) and (6). Furthermore, we define operator B as follows:

$$B : w \to Bw$$

where

$$w = \begin{pmatrix} v_0 \\ D \end{pmatrix}, \quad Bw = \begin{pmatrix} b\frac{\partial u}{\partial \nu}\Big|_\Gamma \\ D + bu \end{pmatrix} \tag{10}$$

and $u,\ v$ are solutions of the problems

$$\begin{aligned} Lv &= D, \quad x \in \Omega, \\ v|_\Gamma &= v_0, \end{aligned} \tag{11}$$

$$\begin{aligned} \Delta u &= v, \ x \in \Omega, \\ u|_\Gamma &= 0. \end{aligned} \tag{12}$$

The operator B primarily defined on couples of smooth functions is extended by continuity on whole $L_2(\Gamma) \times L_2(\Omega)$. Its properties will be investigated later.

For the reduction of BVP (1)–(3) to an equation with the operator, we put

$$u = u_1 + u_2, \quad v = v_1 + v_2,$$

where

$$\begin{aligned} Lv_2 &= f, \quad x \in \Omega, \\ v_2|_\Gamma &= 0, \end{aligned} \tag{13}$$

$$\begin{aligned} \Delta u_2 &= v_2, \quad x \in \Omega, \\ u_2|_\Gamma &= 0, \end{aligned} \tag{14}$$

$$\begin{aligned} Lv_1 &= D, \quad x \in \Omega, \\ v_1|_\Gamma &= v_0, \end{aligned} \tag{15}$$

$$\begin{aligned} \Delta u_1 &= v_1, \quad x \in \Omega, \\ u_1|_\Gamma &= 0 \end{aligned} \tag{16}$$

with L defined in (7).

From (13) and (14), we can determine u_2 and v_2, and from (15) and (16), by the definition of B, we have

$$Bw = \begin{pmatrix} b\frac{\partial u_1}{\partial v}\Big|_\Gamma \\ D + bu_1 \end{pmatrix}. \tag{17}$$

For u to be the solution of (1)–(3), as mentioned above, the relations (3) and (6) should be satisfied, i.e., there should be

$$\frac{\partial u}{\partial v}\Big|_\Gamma = 0, \quad D + bu = 0.$$

From these relations, we derive

$$\begin{aligned} \frac{\partial u_1}{\partial v}\Big|_\Gamma &= -\frac{\partial u_2}{\partial v}\Big|_\Gamma, \\ D + bu_1 &= -bu_2. \end{aligned}$$

Putting

$$F = \begin{pmatrix} -b\frac{\partial u_2}{\partial v}\Big|_\Gamma \\ -bu_2 \end{pmatrix}, \tag{18}$$

from (17), we obtain

$$Bw = F. \tag{19}$$

The smoothness of F depends on that of f, namely, using [5], it is easy to show that if $f \in H^{n-4}(\Omega)$, $n \geq 4$, then $F \in H^{n-3/2}(\Gamma) \times H^n(\Omega)$. Here, as usual, $H^s(\Omega)$, $H^s(\Gamma)$ are Sobolev spaces (see [5]).

Thus, we have led the original problem (1)–(3) to the operator equation (19) for finding the couple $(v_0, D)^T$. After this couple is found, solving the problems (8) and (9), we shall find the solution u of (1)–(3).

Now, let us study the properties of B.

First, we introduce the Hilbert space $H = L_2(\Gamma) \times L_2(\Omega)$ with the scalar product

$$(w, \overline{w})_H = (v_0, \overline{v}_0)_{L_2(\Gamma)} + (D, \overline{D})_{L_2(\Omega)},$$

where

$$w = \begin{pmatrix} v_0 \\ D \end{pmatrix}, \quad \overline{w} = \begin{pmatrix} \overline{v}_0 \\ \overline{D} \end{pmatrix}.$$

Property 1. *B is symmetric in H.*

Proof. We have from (10)

$$(Bw, \overline{w}) = \int_\Gamma \overline{v}_0 b \frac{\partial u}{\partial \nu} d\Gamma + \int_\Omega \overline{D}(D + bu) dx.$$

Transform the boundary integral

$$\begin{aligned} J_1 &= \int_\Gamma \overline{v}_0 b \frac{\partial u}{\partial \nu} d\Gamma = \int_\Gamma \Big(b\overline{v} \frac{\partial u}{\partial \nu} - bu \frac{\partial \overline{v}}{\partial \nu} \Big) d\Gamma \\ &= b \int_\Omega (\overline{v}\Delta u - u\Delta \overline{v}) dx = b \int_\Omega \overline{v} v dx - b \int_\Omega u(a\overline{v} + \overline{D}) dx \\ &= b \int_\Omega \overline{v} v dx - ab \int_\Omega u\overline{v} dx - b \int_\Omega u\overline{D} dx. \end{aligned}$$

Further, we have

$$-ab \int_\Omega u\overline{v} dx = -ab \int_\Omega u\Delta\overline{u} dx = ab \int_\Omega \text{grad} u . \text{grad}\, \overline{u} dx.$$

Thus, we obtain

$$J_1 = b \int_\Omega v\overline{v} dx + ab \int_\Omega \text{grad} u . \text{grad}\, \overline{u} dx - b \int_\Omega u\overline{D} dx.$$

As a result, we obtain

$$(Bw, \overline{w}) = \int_\Omega (bv\overline{v} + ab\, \text{grad} u . \text{grad}\, \overline{u} + D\overline{D}) dx = (B\overline{w}, w). \tag{20}$$

The symmetry of B is proved. ∎

Property 2. *B is positive in H.*

Indeed, from (20), we have

$$(Bw, w) = \int_{\Omega} (bv^2 + ab|\mathrm{grad} u|^2 + D^2)dx \geq 0.$$

The equality occurs if and only if $w = 0$.

Property 3. *B can be decomposed into the sum of a symmetric, positive, completely continuous operator and a projection operator, namely,*

$$B = B_0 + I_2, \tag{21}$$

where B and I_2 are defined as follows:

$$w = \begin{pmatrix} v_0 \\ D \end{pmatrix}, \quad B_0 w = \begin{pmatrix} b\frac{\partial u}{\partial v}\Big|_\Gamma \\ bu \end{pmatrix}, \quad I_2 w = \begin{pmatrix} 0 \\ D \end{pmatrix}, \tag{22}$$

u being defined from (11) and (12).

The complete continuity of B_0 is easily followed from the embedding theorems of Sobolev spaces (see, e.g., [5]). The analogous technique was used in our earlier works [2, 3].

Property 4. *B is bounded in H.*

This fact is a direct corollary of Property 3.

On this occasion, we also consider the perturbed problem obtained from the original problem by replacing the boundary condition (3) with the perturbed one

$$\left(\delta \Delta u + b\frac{\partial u}{\partial v}\right)\Big|_\Gamma = 0. \tag{3'}$$

Here, δ is a small positive parameter.

Doing the same as for (1)–(3), we can reduce the problem (1), (2) and (3′) to the operator equation

$$B_\delta w_\delta = F, \tag{23}$$

where

$$B_\delta = B + \delta I_1, \quad I_1 w_\delta = \begin{pmatrix} v_{\delta 0} \\ 0 \end{pmatrix}, \quad w_\delta = \begin{pmatrix} v_{\delta 0} \\ D_\delta \end{pmatrix} \tag{24}$$

and B, F are defined by (10) and (18).

Taking into account Property 4 of B, we see that B_δ is a linear, symmetric, positive definite operator in H, $B_\delta \geq \delta I$, where I is the identity operator, $I = I_1 + I_2$.

3. Iterative Method for BVP (1)–(3)

Consider the following iterative method for (1)–(3):

(i) given a couple $w^{(0)} = \left(v_0^{(0)}, D^{(0)}\right)^T$;
(ii) knowing $w^{(k)} = \left(v_0^{(k)}, D^{(k)}\right)^T$, $k = 0, 1, \ldots$ solving successively two problems

$$\begin{aligned} Lv^{(k)} &= f + D^{(k)}, \quad x \in \Omega, \\ v^{(k)}|_\Gamma &= v_0^{(k)}, \end{aligned} \tag{25}$$

$$\begin{aligned} \Delta u^{(k)} &= v^{(k)}, x \in \Omega, \\ u^{(k)}|_\Gamma &= 0. \end{aligned} \tag{26}$$

(iii) calculate the new approximation of v_0 and D

$$v_0^{(k+1)} = v_0^k - \tau_{k+1} b \frac{\partial u^{(k)}}{\partial \nu}, \quad x \in \Gamma, \tag{27}$$

$$D^{(k+1)} = D^{(k)} - \tau_{k+1}(D^{(k)} + bu^{(k)}), \quad x \in \Omega, \tag{28}$$

where τ_{k+1} are sufficiently small iterative parameters.

For investigating the convergence of the iterative method, we remark that the above process is a realization of the following iterative scheme:

$$\frac{w^{(k+1)} - w^{(k)}}{\tau_{k+1}} + Bw^{(k)} = F \tag{29}$$

for the operator equation (19).

Indeed, if we represent $u^{(k)} = u_1^{(k)} + u_2$, $v^{(k)} = v_1^{(k)} + v_2$, where u_2, v_2 are the solutions of (13) and (14), then $u_1^{(k)}$, $v_1^{(k)}$ satisfy the following problems:

$$\begin{aligned} Lv_1^{(k)} &= D^{(k)}, \quad x \in \Omega, \\ v_1^{(k)}|_\Gamma &= v_0^{(k)}, \\ \Delta u_1^{(k)} &= v_1^{(k)}, x \in \Omega, \\ u_1^{(k)}|_\Gamma &= 0. \end{aligned} \tag{30}$$

By the definition of B, we have

$$Bw^{(k)} = \begin{pmatrix} b \dfrac{\partial u_1^{(k)}}{\partial \nu}\Big|_\Gamma \\ D + bu_1^{(k)} \end{pmatrix}.$$

Consequently,

$$Bw^{(k)} - F = \begin{pmatrix} b \dfrac{\partial u^{(k)}}{\partial \nu}\Big|_\Gamma \\ D + bu^{(k)} \end{pmatrix}.$$

From here, it is easy to see that (27) and (28) are componentwise writing of (29).

Thus, for studying the convergence of (25)–(28), it suffices to consider (29). Here, it is impossible to use the well-known results of the theory of two-layer iterative schemes for the case of a symmetric, positive, completely continuous operator because operator B is only symmetric and positive but not completely continuous.

Remark. In [1], when condition (4) is satisfied, the authors proposed an iterative method analogous to (25)–(28), but there the convergence is not proved.

In order to construct a convergent iterative process for (1)–(3), we shall use the parametric extrapolation technique, which was developed and used in [2–4].

For the case of observation, we rewrite the perturbed problem associated with the original problem (1)–(3) in the form

$$Au_\delta \equiv \Delta^2 u_\delta - a\Delta u_\delta + bu_\delta = f(x), \quad x \in \Omega, \tag{31}$$

$$u_\delta|_\Gamma = 0, \tag{32}$$

$$\left(\delta \Delta u_\delta + b\frac{\partial u_\delta}{\partial \nu}\right)\Big|_\Gamma = 0. \tag{33}$$

Theorem 1. *Suppose that $f \in H^{n-4}(\Omega)$, $n \geq 4$. Then for the solution of the problem (31)–(33), there holds the following asymptotic expansion:*

$$u_\delta = u + \sum_{i=1}^{N} \delta^i y_i + \delta^{N+1} z_\delta, \quad x \in \Omega, \quad 0 \leq N \leq n - \frac{5}{2} \tag{34}$$

where $y_0 = u$ is the solution of (1)–(3), y_i ($i = 1, ..., N$) are functions independent of δ, $y_i \in H^{n-i}(\Omega)$, $z_\delta \in H^{n-N}(\Omega)$ and

$$\|z_\delta\|_{H^2(\Omega)} \leq C_1, \tag{35}$$

C_1 being independent of δ.

Proof. Under the assumption of the theorem, by [5], there exists a unique solution $u \in H^n(\Omega)$ of the problem (31)–(33). After substituting (34) into (31)–(33) and balancing coefficients of like powers of δ, we see that y_i and z_δ satisfy the following problems:

$$\begin{aligned} Ay_i &= 0, \quad x \in \Omega, \\ y_i|_\Gamma &= 0, \\ b\frac{\partial y_i}{\partial \nu}\Big|_\Gamma &= -\Delta y_{i-1}\big|_\Gamma, \quad i = 1, \ldots, N, \end{aligned} \tag{36}$$

$$\begin{aligned} Az_\delta &= 0, \quad x \in \Omega, \\ z_\delta|_\Gamma &= 0, \\ \left(\delta \Delta z_\delta + b\frac{\partial z_\delta}{\partial \nu}\right)\Big|_\Gamma &= -\Delta y_N\big|_\Gamma. \end{aligned} \tag{37}$$

Once again, using [5], it is not difficult to establish successively that (36) has a unique solution $y_i \in H^{n-i}(\Omega)$ and (37) has a unique solution $z_\delta \in H^{n-N}(\Omega)$.

Clearly, y_i ($i = 1, ..., N$) do not depend on δ. It remains to estimate z_δ. For this purpose, we reduce (37) to a boundary operator equation. We set

$$\Delta z_\delta = v_\delta, \quad D_\delta = -bz_\delta \tag{38}$$

and denote $v_\delta|_\Gamma = v_{\delta 0}$. Then we obtain

$$L v_\delta \equiv \Delta v_\delta - a v_\delta = D_\delta, \quad x \in \Omega,$$
$$v_\delta|_\Gamma = v_{\delta 0},$$
$$\Delta z_\delta = v_\delta, \quad x \in \Omega,$$
$$z_\delta|_\Gamma = 0.$$

Now, denote $w_\delta = \begin{pmatrix} v_{\delta 0} \\ D_\delta \end{pmatrix}$. Then by definition

$$B w_\delta = \begin{pmatrix} b \frac{\partial z_\delta}{\partial \nu}\Big|_\Gamma \\ D_\delta + b z_\delta \end{pmatrix}.$$

Using the last condition of (37), we obtain

$$B w_\delta + \delta I_1 w_\delta = h, \tag{39}$$

where

$$h = \begin{pmatrix} -\Delta y_N \\ 0 \end{pmatrix}.$$

It is possible to verify that (see [4])

$$(B w_\delta, w_\delta) \le (B w, w), \tag{40}$$

where w is the solution of the equation $Bw = h$. This equation has a solution because it is the equation to which the problem (37) with $\delta = 0$ may be reduced.

In Sec. 2, when investigating the properties of B, we have established that

$$(B w_\delta, w_\delta) = \int_\Omega (b v_\delta^2 + ab|\operatorname{grad} z_\delta|^2 + D_\delta^2) dx.$$

In view of (38), we have

$$(B w_\delta, w_\delta) = \int_\Omega (b|\Delta z_\delta|^2 + ab|\operatorname{grad} z_\delta|^2 + b^2 |z_\delta|^2) dx.$$

Since the right-hand side of the above equality defines a norm in the class of functions vanishing on the boundary, which is equivalent to the norm $\|\cdot\|_{H^2(\Omega)}$, we have

$$\|z_\delta\|_{H^2(\Omega)} \le C_1,$$

where $C_1 = C_2\sqrt{(Bw, w)}$, C_2 being independent of δ.

Thus, the theorem is proved. ■

As usual (see [2–4]), we construct an approximate solution U^E of the original problem (1)–(3) by the formula

$$U^E = \sum_{i=1}^{N+1} \gamma_i u_{\delta/i},$$

where

$$\gamma_i = \frac{(-1)^{N+1-i}\, i^{N+1}}{i!\,(N+1-i)!},$$

where $u_{\delta/i}$ is the solution of (31)–(33) with the parameter $\delta\,/\,i$ ($i = 1, \ldots, N+1$). Then it is easy to obtain the following estimate:

$$\|U^E - u\|_{H^2(\Omega)} \leq C_2\delta^{N+1},$$

where u is the solution of (1)–(3) and C_2 a constant independent of δ.

For solving (31)–(33), which may be reduced to Eq. (23), we propose to use the following iterative process under the assumption $f \in L_2(\Omega)$:

(i) given a couple $(v_{\delta 0}^{(0)}, D_\delta^{(0)}) \in H^{1/2}(\Gamma) \times L_2(\Omega)$;

(ii) knowing $v_{\delta 0}^{(k)}$ and $D_\delta^{(k)}$, $k = 0, 1, \ldots$ solves successively two problems

$$\begin{aligned} Lv_\delta^{(k)} &= f + D_\delta^{(k)}, \ x \in \Omega, \\ v_\delta^{(k)}|_\Gamma &= v_{\delta 0}^{(k)}, \\ \Delta u_\delta^{(k)} &= v_\delta^{(k)}, \quad x \in \Omega, \\ u_\delta^{(k)}|_\Gamma &= 0; \end{aligned}$$

(iii) calculate the new approximation of v_0 and D

$$\begin{aligned} v_{\delta 0}^{(k+1)} &= v_{\delta 0}^{(k)} - \tau_{\delta,k+1}\left(b\frac{\partial u_\delta^{(k)}}{\partial \nu} + \delta v_{\delta 0}^{(k)}\right), \quad x \in \Gamma, \\ D_\delta^{(k+1)} &= D_\delta^{(k)} - \tau_{\delta,k+1}(D_\delta^{(k)} + bu_\delta^{(k)}), \quad x \in \Omega, \end{aligned}$$

where $\{\tau_{k+1}\}$ is the Chebyshev collection of parameters according to bounds $\gamma_\delta^{(1)} = \delta$, $\gamma_\delta^{(2)} = \delta + \|B\|$ (see [2–4, 6] for details).

In the case of simple iteration

$$\tau_{\delta,k} \equiv \tau_{\delta,0} = \frac{2}{\gamma_\delta^{(1)} + \gamma_\delta^{(2)}},$$

we obtain

$$\left\|w_\delta^{(k)} - w_\delta\right\|_H \leq (\rho_\delta)^k \left\|w_\delta^{(0)} - w_\delta\right\|_H, \tag{41}$$

where

$$w_\delta^{(k)} = \begin{pmatrix} w_{\delta 0}^{(k)} \\ D_\delta^{(k)} \end{pmatrix}, \quad w_\delta = \begin{pmatrix} v_{\delta 0} \\ D_\delta \end{pmatrix}$$

and

$$\rho_\delta = \frac{1 - \xi_\delta}{1 + \xi_\delta}, \quad \xi_\delta = \frac{\gamma_\delta^1}{\gamma_\delta^2}.$$

This result follows from the general theory of two-layer iterative schemes [6] applied to the operator equation (23), which is obtained from (31)–(33).

Using estimates for the solution of elliptic problems [5] and taking into account (41), we obtain the estimate

$$\left\| u_\delta^{(k)} - u_\delta \right\|_{H^{5/2}(\Omega)} \leq C(\rho_\delta)^k \left\| w_\delta^{(0)} - w_\delta \right\|_{H},$$

where C is a constant independent of δ.

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