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Some Characterizations of V-modules and Rings

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Abstract. A module M has the property (V) if, for every $K \leq M$, $K \neq M$ and $m \in M - K$, any submodule L maximal, with respect to the property that it contains K but does not contain the element m, is maximal in M. It has the property (Ve) if (V) holds for every essential proper submodule K and $m \in M - K$. It is shown that M is a V-module if and only if M has the property (V). $M/\mathrm{Soc}M$ is a V-module if and only if M has the property (Ve). Some further characterizations of V-rings and GV-rings are given.

All rings considered are associative and have an identity, and all modules are unitary right modules. Let R be a ring and M a module. We write Rad$M$, $\mathbf{Z}(M)$, Soc$M$ and E(M) for the radical, the singular submodule, the socle and the injective envelope of M, respectively. Let M and N be modules. N is called M-injective if, for each submodule K of M, every homomorphism from K into N can be extended to an R-homomorphism from M into N. M is called a V*-module* by Hirano [6] (or *cosemisimple* by Fuller [2]). Every proper submodule of M is an intersection of maximal submodules. R is called a V*-ring* if the right module R_R is a V-module. M is a V-module if and only if every simple module is M-injective. Following Hirano [6], M is called a *generalized V-module* or a *GV-module* if every simple singular module is M-injective. If the module R_R is a GV-module, R is called a GV-ring.

In this note we give some characterizations of V-modules and GV-modules in terms of certain maximal submodules.

We write $N \leq M$ if N is a submodule of M. A right R-module M is said to have property (V) and (Ve), respectively, if

(V) *for every $K \leq M$, $K \neq M$ and $m \in M - K$, any submodule L maximal, with respect to the property that it contains K but does not contain, the element m is maximal in M;*

(Ve) *(V) holds for every essential proper submodule K and $m \in M - K$.*

1. Modules with Properties (V) and (Ve)

In [8], it is proved that R is a right V-ring if and only if the right R-module R has the property (V).

Theorem 1. *Let M be a module. Then the following are equivalent:*

(1) *M is a V-module;*
(2) *M has the property (V).*

Proof. (1) $\Rightarrow$ (2) Let $K \leq M$, $K \neq M$, $m \in M - K$ and let L be a maximal submodule with respect to the property that L contains K but does not contain the element m. Then $(mR+L)/L$ is a simple R-module. By (1), it is M-injective and so M/L-injective. Also, $(MR+L)/L$ is an essential submodule of M/L. Hence, $(mR+L)/L = M/L$. Thus, L is a maximal submodule of M.

(2) $\Rightarrow$ (1) Let X be a simple module, N an essential proper submodule of M, f a non-zero homomorphism from N to X and let $x \in N - \ker f$. Let L be a submodule of M maximal with respect to $x \notin L$ and $\ker f \leq L$. Then $xR + L = M = N + L$ by (2). Hence $N \cap L$ is maximal in N. Since $\ker f$ is a maximal submodule of N, then $N \cap L = \ker f$. Thus f extends to M.

Theorem 2. *Let M be a module. Then the following are equivalent:*

(1) *$M/\mathrm{Soc}M$ is a V-module;*
(2) *M has the property (Ve).*

Proof. (1) $\Rightarrow$ (2). Let $m \in M$ and let N be an essential submodule of M maximal with respect to $m \notin N$. Then $(mR+N)/N$ is a simple module and essential in M/N. By (1) it is $M/\mathrm{Soc}M$-injective. Since $\mathrm{Soc}M \leq N$, then $(mR+N)/N$ is M/N-injective. Thus $M = mR + N$. This implies that N is a maximal submodule of M.

(2) $\Rightarrow$ (1). Let X be any simple module. To prove X is $M/\mathrm{Soc}M$-injective, let $N/\mathrm{Soc}M$ be an essential submodule of $M/\mathrm{Soc}M$ and f a non-zero homomorphism from $N/\mathrm{Soc}M$ to X. Set $\mathrm{Ker} f = K/\mathrm{Soc}M$ for some $K \leq M$. Then N is an essential submodule of M and K is a maximal submodule of N. We consider two cases: Assume K is essential in N. Then K is essential in M. Let $x \in N - K$ and let L be a submodule of M maximal with respect to $x \notin L$ and $K \leq L$. Since K is essential in M, then L is essential in M. By (Ve), L is a maximal submodule of M, and so $M = xR + L = N + L$. Then $N \cap L$ is maximal in N. Hence, $K = N \cap L$. Thus, $K/\mathrm{Soc}M = (N/\mathrm{Soc}M) \cap (L/\mathrm{Soc}M)$, which is the kernel of f. It follows that f extends to a homomorphism from $M/\mathrm{Soc}M$ to X. If K is not essential in N, then K is a direct summand of N and $N = K \oplus T$ for some $T \leq N$. Hence, N/K, T and X are isomorphic simple modules. It follows that $T \leq \mathrm{Soc}M$. Since $\mathrm{Soc}M \leq K$, then $T = 0$. This is a contradiction which completes the proof.

2. Co-singular Submodule Z*(M) and V-rings

A submodule N of M is called *small* in M such that, whenever $N + L = M$ for some submodule L of M, we have $M = L$. A module M is said to be small if M is small in E(M) [7]. Let M be an R-module. We set $\mathbf{Z}^*(M) = \{m \in M : mR \text{ is small}\}$. We call $\mathbf{Z}^*(M)$ a co-singular submodule of M. In this note, we consider the classes $\underline{X} = \{R\text{-module } M : \mathbf{Z}^*(M) = 0\}$ and $\underline{X}^* = \{R\text{-module } M$: whenever $Q \leq P \leq M$, $P/Q \in X$ implies $P/Q = 0\}$ following [5]. Submodules and homomorphic images of small modules are small [7] and $\underline{X}$ is closed under submodules, direct products, direct sums, essential extensions and module extensions. $\underline{X}^*$ is closed under submodules, homomorphic images and direct sums. Any member of $\underline{X}$ is called an $\underline{X}$-module. $\underline{X} \cap \underline{X}^* = 0$ and since Rad$M$ is the sum of all small submodules of M, Rad$M \leq \mathbf{Z}^*(M)$ and $\mathbf{Z}^*(M) = M \cap \operatorname{Rad} E(M)$. $\mathbf{Z}^*(E) = \operatorname{Rad} E$ for any injective module E. In general, $\mathbf{Z}^*(M) \neq \operatorname{Rad} M$ (e.g., Example 11).

Lemma 3. *Let M be a module and $N \leq M$. Then $(\mathbf{Z}^*(M) + N)/N$ is a submodule of $\mathbf{Z}^*(M/N)$.*

Proof. Let $m \in \mathbf{Z}^*(M)$. Then mR is small in E(mR) so that $(mR + N)/N$ is small in $(\mathrm{E}(mR) + N)/N$. Hence, $(m + N)R = (mR + N)/N$ is small. Thus, $m + N \in \mathbf{Z}^*(M/N)$. ■

Lemma 4. *Let M be a module. Then*

(1) *if M is small then $\mathbf{Z}^*(M) = M$;*
(2) *if $\mathbf{Z}^*(M) = M$ then $M \in \underline{X}^*$;*
(3) *if M is semisimple injective, then $M \in \underline{X}$.*

Proof. (1) It is clear from the definitions.

(2) Let M be a module and $Q \leq P \leq M$ be such that $\mathbf{Z}^*(M) = M$ and $P/Q \in \underline{X}$. Let $x \in P$. Then xR and $(xR + Q)/Q$ are small and $(xR + Q)/Q \in \underline{X}$. By (1), $(xR + Q)/Q \in \underline{X}^*$. Hence, $xR + Q = Q$ and $x \in Q$. Thus, $M \in \underline{X}^*$.

(3) Assume M is semisimple injective. Since $\underline{X}$ is closed under direct sums, without loss of generality, we may assume M is simple injective. If $\mathbf{Z}^*(M) = M$, then M is small in M. This is a contradiction. Hence $\mathbf{Z}^*(M) = 0$ and so $M \in \underline{X}$. This completes the proof. ■

Lemma 5. *For any module M, $\mathbf{Z}^*(M) = 0$ if and only if $\operatorname{Rad} E(M) = 0$.*

Proof. M is essential in E(M). ■

Proposition 6. *Let R be a ring such that $R/J(R)$ is right Artinian. Then $\mathbf{Z}^*(M) = 0$ if and only if M is semisimple injective.*

Proof. Sufficiency is clear from Lemma 4(3). Conversely, suppose $\mathbf{Z}^*(M) = 0$. Then $0 = \operatorname{Rad} \mathrm{E}(M) = \mathrm{E}(M)\,\mathrm{J}(R)$. Hence, E($M$) is semisimple and so $M = \mathrm{E}(M)$. Thus, M is semisimple injective. ■

Example 7. Let R be a prime right Goldie ring which is not right primitive (e.g., a commutative domain which is not a field). Then $\mathbf{Z}^*(R) = R$.

Proof. Let $r \in R$ and $E = \mathrm{E}(rR)$. Suppose $E = rR + L$ for some $L \leq E$. If r is not in L, then E/L is nonzero and a cyclic module so that there exists a maximal submodule P of E with L contained in P. The module $U = E/P$ is simple, and if I is its annihilator in R, we know that I is a nonzero ideal of R by our hypothesis. But in this case, I contains a nonzero divisor by Goldie's Theorem [4, Proposition 5.9] and then $E = EI$ by [9, Proposition 2.6] so that $E = P$, a contradiction. Hence, $r \in L$ and so $E = L$ and rR is small. Thus, $\mathbf{Z}^*(R) = R$.

Lemma 8. *Let R be a ring such that $\mathbf{Z}^*(R) = R$. Then, for every module M, $\mathbf{Z}^*(M) = M$.*

Proof. Let M be a module and $m \in M$. Let $r(m)$ denote the right annihilator of m in R. Then $mR \cong R/r(m)$ and $\mathbf{Z}^*(R) = R$ imply that mR is small and so $m \in \mathbf{Z}^*(M)$. ■

We combine Example 7 and Lemma 8.

Corollary 9. *Let R be a prime right Goldie ring which is not a right primitive ring. Then, for every module M, $\mathbf{Z}^*(M) = M$.*

Theorem 10. *Let R be a ring. Then the following are equivalent:*

(1) *R is a right GV-ring;*
(2) *every $\underline{X}^*$-module is projective;*
(3) *every simple $\underline{X}^*$-module is projective;*
(4) *for every R-module M with $\mathbf{Z}^*(M) \neq 0$, $\mathbf{Z}^*(M)$ is projective;*
(5) *every small module is projective;*
(6) *for every R-module M with $\mathbf{Z}^*(M) = M$, M contains a nonzero projective submodule;*
(7) *for every R-module M, $\mathbf{Z}(M) \cap \mathbf{Z}^*(M) = 0$;*
(8) *for every right ideal I of R, $\mathbf{Z}(R/I) \cap \mathbf{Z}^*(R/I) = 0$;*
(9) *for every R-module M with $\mathbf{Z}(M)$ essential in M, $\mathbf{Z}^*(M) = 0$;*
(10) *$R/\mathrm{Soc}R$ is a V-module and $\mathbf{Z}(R) \cap \mathbf{Z}^*(R) = 0$;*
(11) *every proper essential right ideal of R is an intersection of maximal right ideals and $\mathbf{Z}(R) \cap \mathbf{Z}^*(R) = 0$;*
(12) *for every essential right ideal K of R, $\mathbf{Z}^*(R/K) = 0$ and $\mathbf{Z}(R) \cap \mathbf{Z}^*(R) = 0$.*

Proof. (1) ⇒ (2). Let $M \in \underline{X}^*$ and $m \in M$, $m \neq 0$. Let K be a maximal submodule of mR. Then mR/K is injective or projective. If mR/K is injective, then by Lemma 4(3), $mR/K \in X$. Hence, $mR/K = 0$. Thus, it is projective. It follows that K is a direct summand of mR, and so mR is semisimple and so is M. As before, it can be shown that every simple submodule of M is projective.

(2) ⇒ (3). This is clear.

(3) ⇒ (4). Since $\mathbf{Z}^*(M)$ is in $\underline{X}^*$, by Lemma 4(2), every simple module is injective or small; the proof is the same as that of (1 ⇒ 2).

(4) ⇒ (5). Let M be a nonzero small module. Then $\mathbf{Z}^*(M) = M$ by Lemma 4(1). Thus, M is projective by (4).

(5) ⇒ (6). Let M be a module with $\mathbf{Z}^*(M) = M$. Let $m \in M$, $m \neq 0$. Since mR is small, then mR is projective by (5).

(6) ⇒ (7). Let $m \in \mathbf{Z}(M) \cap \mathbf{Z}^*(M)$. Then $\mathbf{Z}^*(mR) = mR$. Assume $m \neq 0$. Then by (6), mR contains a nonzero projective submodule L. Hence, L is isomorphic to $I/r(m)$ for some right ideal I of R. Thus, $r(m)$ is a direct summand of I. But since $m \in \mathbf{Z}(M)$, $r(m)$ is essential in R, and therefore, in I, $L = 0$, a contradiction.

(7) ⇒ (8). This is clear.

(8) ⇒ (9). Let M be a module with $\mathbf{Z}(M)$ essential in M. Let $x \in \mathbf{Z}^*(M)$. Assume $x \neq 0$. There exists a nonzero $m \in xR \cap \mathbf{Z}(M)$. Then $mR \leq \mathbf{Z}^*(M) \cap \mathbf{Z}(M)$. Hence, $mR \cong R/r(m) \leq \mathbf{Z}^*(R/r(m)) \cap \mathbf{Z}(R/r(m))$ which is zero by (8). This is a contradiction.

(9) ⇒ (10). Let X be a simple module, $I/\mathrm{Soc}R$ a right ideal of $R/\mathrm{Soc}R$ and f a nonzero homomorphism from $I/\mathrm{Soc}R$ to X. Set $\mathrm{Ker} f = K/\mathrm{Soc}R$ for some right ideal K of R. Then K is a maximal right ideal of I. If K is not essential in I, then $I = K \oplus T$ for some $T \leq I$. Hence, $T \leq \mathrm{Soc}R \leq K$. This is a contradiction. It follows that K is essential in I, and so I/K is singular. By (9), $\mathbf{Z}^*(I/K) = 0$ and then $\mathbf{Z}^*(X) = 0$. Since X is simple, then X is injective and is therefore $R/\mathrm{Soc}R$-injective. It follows that f extends to $R/\mathrm{Soc}R$.

(10) ⇒ (11). $R/\mathrm{Soc}R$ is a V-module if and only if every proper essential right ideal of R is an intersection of maximal right ideal [10].

(11) ⇒ (12). Let K be an essential right ideal of R. Let $0 \neq x + K \in \mathbf{Z}^*(R/K)$. By (11), there exists a maximal right ideal L of R such that $x \notin L$ and $K \leq L$. Then $(xR + L)/L$ is small and a singular module. Next, we prove $(xR + L)/L$ is an injective module. Let I be an essential right ideal of R and f a nonzero homomorphism from I to $(xR + L)/L$. Set $T = \mathrm{Ker} f$. Assume T is essential in I. Then T is an essential right ideal in R. By (11), we may find a maximal right ideal J of R so that $T \leq J$ and $I \not\leq J$. Hence, $R = I + J$. Since $T \leq I \cap J \leq I$ and $I \not\leq J$, therefore, $T = I \cap J$, and so f extends. If T is not an essential right ideal in I, then $I = T \oplus U$ for some right ideal U of R. Hence, U is a simple singular and a small module. Thus, $U \leq \mathbf{Z}(R) \cap \mathbf{Z}^*(R)$ that is zero. This is a contradiction for f is a nonzero mapping. It follows that $(xR + L)/L$ is an injective module. This is a contradiction because $(xR + L)/L$ is a small module. Hence, $\mathbf{Z}^*(R/K) = 0$.

(12) ⇒ (1). Let X be a simple singular module and I an essential right ideal of R. Let f be a nonzero homomorphism from I to X with kernel K. Then K is a maximal submodule of I. If K is not essential in I, then $I = K \oplus L$ for some $L \leq I$. Then L is a simple singular right ideal of R. Hence, $L^2 = 0$ or $L = eR$ for some idempotent e of R. Assume $L = eR$. Then $r(e) = (1-e)R$ is essential in R. This is a contradiction. Hence, $L^2 = 0$ and so $L \leq \mathrm{Rad}R$. Since L is singular then $L \leq \mathbf{Z}(R)$. Since $\mathrm{Rad}R \leq \mathbf{Z}^*(R)$, then by (12), $L = 0$. Hence, K is essential in I and thus in R too. By (12), $\mathbf{Z}^*(R/K) = 0$ and so $\mathbf{Z}^*(I/K) = 0$. This and I/K simple imply that I/K is injective. Since $X \cong I/K$, X is injective. This completes the proof. ∎

Example 11. Let $R = \begin{vmatrix} F & 0 \\ F & F \end{vmatrix}$ be lower triangular matrices over a field F. $J(R) = \begin{vmatrix} 0 & 0 \\ F & 0 \end{vmatrix}$, $\mathrm{Soc}(R_R) = \begin{vmatrix} F & 0 \\ F & 0 \end{vmatrix}$ and by [1, Example 4.b], R is a right and left GV-ring and not a V-ring. $\mathbf{Z}^*(R)$ is semisimple by the proof of Theorem 10 (1 ⇒ 2) and $J(R) \leq \mathbf{Z}^*(R) \leq \mathrm{Soc}R$. Set $K = \begin{vmatrix} 0 & 0 \\ F & F \end{vmatrix}$ and $L = \begin{vmatrix} F & 0 \\ 0 & 0 \end{vmatrix}$. By [3, Exercise 3.B.20–21] K is an injective right ideal and every injective right ideal of R is contained in K. Since the simple right ideal L is injective or small and L is not in K, then L is small. Hence, $\mathbf{Z}^*(R) = \mathrm{Soc}(R_R)$ and $\mathrm{J}(R) \neq \mathbf{Z}^*(R)$.

Theorem 12. *Let R be a ring. Then the following are equivalent:*

(1) *R is a right V-ring;*
(2) *for every R-module M, $\mathbf{Z}^*(M) = 0$;*
(3) *for every simple R-module M, $\mathbf{Z}^*(M) = 0$.*

Proof. (1) $\Rightarrow$ (2). By (1), $\mathrm{Rad}\,\mathrm{E}(M) = 0$. Hence, $\mathbf{Z}^*(M) = 0$.

(2) $\Rightarrow$ (3). This is clear.

(3) $\Rightarrow$ (1). Let M be a simple module. By (3), $\mathbf{Z}^*(M) = 0$. Since M is simple, M is injective or small. Assume M is small, then by Lemma 4, $M \in \underline{X}^*$. This is a contradiction. Hence, M is injective. ■

We combine Theorems 1 and 12.

Corollary 13. *Let R be a ring. Then R_R has the property (V) if and only if $\mathbf{Z}^*(M) = 0$ for every R-module M.*

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