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Holomorphic Dirichlet Series in the Half Plane*

Lê Hai Khôi

Institute of Information Technology,
Nghia Do, Cau Giay, 10000 Hanoi, Vietnam

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Abstract. This paper deals with Dirichlet series with real frequencies that define holomorphic functions in the half plane of **C**.

1. Introduction

Dirichlet series with real frequencies which represent entire functions on the complex plane **C** have been investigated by many authors. Several problems such as topological structures, linear continuous functionals, bases, etc., have been considered. However, there seems to be few papers on non-entire (holomorphic) Dirichlet series with real frequencies. In this work, we are concerned with the last series.

Given a non-entire Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z},$$

where $a_n \in \mathbf{C}$ and $0 < (\lambda_n) \uparrow +\infty$. As is well known, there exists a number R_c called the abscissa of convergence, such that the sum of the series converges at all z with $\mathrm{Re}z > R_c$ and disconverges at all z with $\mathrm{Re}z < R_c$. The abscissa of absolute convergence R_a is defined similarly. Between these numbers, there is a relation:

$$0 \leq R_a - R_c \leq L = \limsup_{n\to\infty} \frac{\log n}{\lambda_n}.$$

In the case $L = 0$, the abscissa of convergence and the abscissa of absolute convergence of Dirichlet series coincide and can be defined by the formula:

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$$R_c = R_a = \limsup_{n\to\infty} \frac{\log |c_n|}{\lambda_n}.$$

This paper deals with Dirichlet series with real frequencies which represent holomorphic functions in the half plane of $\mathbf{C}$. We describe briefly the content of our work. Section 2 deals with some auxiliary results concerning the convergence of Dirichlet series in a space of holomorphic functions in the half plane. These results are obtained in the spirit of [4, 7] for the case of holomorphic Dirichlet series with complex frequencies in bounded convex domains of $\mathbf{C}^n$, $n \geq 1$. In Sec. 3, we study a sequence space of the coefficients of Dirichlet series. Here, we follow the terminology in [5, 9]. We endow it, as was the case for entire Dirichlet series, with some topological structures and compare them. In particular, we show that the picture in the holomorphic case is quite different from the entire one. Section 4 concerns various dualities of the sequence space introduced in the previous section, namely, we study coefficient multipliers between spaces E_R and l^p $(0 < p \leq \infty)$.

2. Sequence Space of Coefficients

Consider Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, \tag{2.1}$$

with $0 < (\lambda_n) \uparrow +\infty$.

Note that, since the sequence of frequencies is real, the series (2.1) has the uniqueness of representation, i.e., different sequences of coefficients (a_n) represent different functions. Due to this fact, we can always identify Dirichlet series (2.1) with the sequence (a_n) of its Dirichlet coefficients.

Let R be a given real number. First, we make a characterization of the coefficients of the series (2.1) when it converges for the topology of $\mathcal{O}(\Pi_R)$, the space of holomorphic functions in the half plane Π_R with the usual topology of uniform convergence on compact subsets of Π_R, where $\Pi_R = \{z;\ \mathrm{Re} z > R\}$.

Theorem 2.1. *If the multiple Dirichlet series (2.1) converges for the topology of $\mathcal{O}(\Pi_R)$ and $\lambda_n \to \infty$ as $n \to \infty$, then*

$$\limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n} \leq R. \tag{2.2}$$

Conversely, if the coefficients of (2.1) satisfy condition (2.2) and, in addition, the sequence (λ_n) satisfies the condition

$$\lim_{n\to\infty} \frac{\log n}{\lambda_n} = 0, \tag{2.3}$$

then the series (2.1) converges absolutely for the topology of $\mathcal{O}(\Pi_R)$.

Proof. We consider a family (K_q) of compact subsets of the half plane Π_R of the form

$$K_q = \left\{z;\ R + \frac{1}{q} \leq \text{Re}z \leq R + q,\ |\text{Im}z| \leq q\right\},\ 0 < q \uparrow +\infty. \tag{2.4}$$

Necessity. Suppose the series (2.1) converges for the topology of $\mathcal{O}(\Pi_R)$. Then for any $q \in (0, +\infty)$, there exists a positive constant $M_q < \infty$ such that

$$\sup\left\{\left|a_n e^{-\lambda_n z}\right|;\ z \in K_q, n \geq 1\right\} \leq M_q,$$

which, in view of (2.4), is equivalent to

$$\frac{\log|a_n|}{\lambda_n} \leq \frac{\log M_q}{\lambda_n} + R + \frac{1}{q},\ \forall n \geq 1.$$

Consequently,

$$\limsup_{n\to\infty} \frac{\log|a_n|}{\lambda_n} \leq R + \frac{1}{q}.$$

Letting q tend to $+\infty$, we obtain the inequality (2.2).

Sufficiency. Let conditions (2.2) and (2.3) hold. Take an arbitrary compact subset K of Π_R. Then it is clear that $K \subset K_q$ for some $q \in (0, +\infty)$, where K_q is defined by (2.4). We shall prove that

$$\sum_{n=1}^{\infty} |a_n| e^{-\lambda_n (R+\frac{1}{q})} < \infty.$$

By (2.2), for $0 < \varepsilon < 1/q$, there exists N_1 such that $\forall n > N_1$

$$\frac{\log|a_n|}{\lambda_n} \leq R + \varepsilon,$$

or

$$|a_n| < e^{\lambda_n (R+\varepsilon)}.$$

Hence, for $n > N_1$

$$|a_n| e^{-\lambda_n (R+\frac{1}{q})} \leq e^{(\varepsilon - \frac{1}{q})\lambda_n}.$$

By (2.3), there exists N_2 such that $\forall n > N_2$

$$2\log n < \lambda_n (\frac{1}{q} - \varepsilon),$$

or

$$e^{(\varepsilon - \frac{1}{q})\lambda_n} < \frac{1}{n^2}.$$

Therefore, $\forall n > \max(N_1, N_2)$

$$|a_n| e^{-\lambda_n (R+\frac{1}{q})} < \frac{1}{n^2}.$$

So, we obtain

$$\sum_{n=1}^{\infty} |a_n| e^{-\lambda_n (R+\frac{1}{q})} < \infty,$$

which means that the series (2.1) converges absolutely for the topology of $\mathcal{O}(\Pi_R)$. ∎

Corollary 2.2. *If (2.3) holds, then the series (2.1) converges for the topology of $\mathcal{O}(\Pi_R)$ if and only if it converges absolutely for the topology of $\mathcal{O}(\Pi_R)$.*

Remark 2.3. When the frequencies (λ_n) are complex numbers, the domain of absolute convergence of Dirichlet series, as is well known, is convex. In this case, this result was proved for a bounded convex domain in [4] for one variable and in [7] for several variables.

Denote by $\mathcal{E}(\Pi_R)$ the class of functions f of the form

$$f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z},$$

where (a_n) satisfy condition (2.2). We see that if $f \in \mathcal{E}(\Pi_R)$, then $f \in \mathcal{O}(\Pi_R)$, and moreover, the class $\mathcal{E}(\Pi_R)$ contains all entire functions represented by Dirichlet series with frequencies (λ_n). It is easy to verify that it forms a vector space with usual pointwise addition and scalar multiplication.

As was noted above, due to the uniqueness of the representation of Dirichlet series with real frequencies, we can indentify the class $\mathcal{E}(\Pi_R)$ with the class E_R of their Dirichlet coefficients. Thus,

$$E_R = \{(a_n) \text{ satisfies } (2.2)\}.$$

It is obvious that E_R is a vector space called a Dirichlet sequence space.

In the sequel, we study various properties of the space E_R.

3. Topological Structures

We shall show that the sequence space E_R can be endowed with some topological structure. Before doing so, we would like to introduce the terminology we adopt in the present paper: a Fréchet space is any metrizable and complete locally convex topological vector space; an (F)-space is metrizable and complete, but not necessarily locally convex.

A natural question arises: Is it possible to endow the space E_R with some topological structure? Below, we see that there are different ways to do this.

So far, this problem was first considered by Kamthan and Shing Gautam in [3], namely, for each $a = (a_n) \in E_R$, it can define norms on E_R as follows:

$$\|a\|_k = \sum_{n=1}^{\infty} |a_n| e^{-\sigma_k \lambda_n}, \ (\sigma_k) \downarrow R.$$

Denote by τ_σ the topology on E_R generated by the family of norms $(\|\cdot\|_k)$. Then (E_R, τ_σ) is a Fréchet space (i.e., the complete, metrizable locally convex space).

Furthermore, consider the family of pseudo-norms defining the compact-open topology of the space $\mathcal{O}(\Pi_R)$. As a sequence of compact subsets converging to Π_R from inside, we can take (K_q) of the compact subsets of the half plane Π_R of the form (2.4), i.e.,

$$K_q = \left\{z;\ R + \frac{1}{q} \le \mathrm{Re} z \le R + q,\ |\mathrm{Im} z| \le q\right\},\ q = 1, 2, \ldots .$$

Then for each $a = (a_n) \in E_R$, we denote

$$|a|_q = \sum_{n=1}^{\infty} \sup\left\{|a_n e^{-\lambda_n z}|;\ z \in K_q\right\} = \sum_{n=1}^{\infty} |a_n| e^{-(R+\frac{1}{q})\lambda_n},\ q \geq 1.$$

It is easy to see that $|\cdot|_q$ is a pseudo-norm on the space E_R. Denote by τ_0 the topology generated by $\{|\cdot|_q\}_{q=1}^{\infty}$. Taking into account (2.4), it is easy to verify that the topology τ_0 is equivalent to the topology τ_σ. So we have the following result.

Proposition 3.1. *In the space E_R, two topologies τ_σ and τ_0 are equivalent.*

On the other hand, for each $a = (a_n) \in E_R$, we can also define the following function:

$$\|a\|_E = \sup_{n\geq 1}\left\{|a_n|^{1/\lambda_n}\right\}. \tag{3.1}$$

Due to (2.2), function (3.1) is well defined and it is a paranorm (see, e.g., [5, 9]) on E_R. Denote by τ the topology given by $\|\cdot\|_E$.

Before continuing, we make the following note. It is well known that for entire Dirichlet series (with real frequencies), i.e., for the case $R = -\infty$, two topologies τ_σ and τ are equivalent, which means that the space (E_R, τ) is also a Fréchet space (see, e.g., [3]). However, for Dirichlet series with complex frequencies, the picture is quite different. In this case, the topology τ is no more locally convex and it is strictly stronger than topology τ_σ (see [8]).

It can be asked: What about our case of non-entire (holomorphic) Dirichlet series? We are concerned with this question.

Denote

$$\rho(a,b) = \|a - b\|_E = \sup_{n\geq 1}\left\{|a_n - b_n|^{1/\lambda_n}\right\},\ a = (a_n), b = (b_n) \in E_R.$$

It is easy to verify that $\rho(a, b)$ is an invariant metric on E_R. As it was in [7], we can prove the following result.

Theorem 3.2. (E_R, τ) *is a complete metrizable, non-locally bounded space, i.e., a non-normable (F)-space.*

Furthermore, we have the following result.

Proposition 3.3. *In the space E_R, the topology defined by the metric ρ is not locally convex. In other words, this space with the metric ρ is never a Fréchet space. Moreover, the topology τ is strictly stronger than the topology τ_σ.*

Proof. Suppose $(a^{(j)}) \subset E_R$ and $a^{(j)} \to 0$ with respect to the topology τ. We show that the sequence $(a^{(j)})$ tends to 0 under each norm $\|\cdot\|_k$, $k = 1, 2, \ldots$. Take an arbitrary number $k \in \mathbf{N}$ and let $\varepsilon > 0$ be given. Then for $s \in \mathbf{N}$ with $\pi^2 e^{-\lambda_1 \log s} < 6\varepsilon$, there exists $N_1 = N_1(s)$ such that $\forall j \geq N_1$

$$\|a^{(j)}\| = \sup_{n\geq 1}\left|a_n^{(j)}\right|^{1/\lambda_n} < \frac{1}{s} = e^{-\log s}. \tag{3.2}$$

On the other hand, by condition (2.3), there is a number $N_2 = N_2(s)$ such that $\forall n \geq N_2$. We have

$$\lambda_n > \frac{\lambda_1 \log s + 2\log n}{\sigma_k + \log s},$$

which implies that

$$\lambda_1 \log s + 2\log n < \lambda_n(\sigma_k + \log s). \tag{3.3}$$

Combining (3.2)–(3.3), we have that $\forall j \geq N = \max\{N_1, N_2\}$

$$\|a^{(j)}\|_k = \sum_{n=1}^{\infty} |a_n^{(j)}| e^{-\sigma_k \lambda_n} \leq \sum_{n=1}^{\infty} e^{-\lambda_n(\log s + \sigma_k)}$$

$$\leq e^{-\lambda_1 \log s} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} e^{-\lambda_1 \log s} < \varepsilon.$$

So we have proved that in the space E_R, the convergence of a sequence with respect to the metric ρ implies its convergence with respect to the topology τ_σ.

Now, we assume $a^{(j)} \to a$ under each norm $\|\cdot\|_k$, $k = 1, 2, \ldots$. We show that in general, the sequence $(a^{(j)})$ need not tend to a with respect to the topology τ.

Take an arbitrary element $a = (a_n)$ of the space E_R. Then for each $\sigma > R$, the series $\sum_{n=1}^{\infty} a_n e^{-\sigma\lambda_n}$ converges, which implies

$$\sum_{n=j+1}^{\infty} |a_n| e^{-\sigma\lambda_n} \to 0, \ j \to \infty. \tag{3.4}$$

Consider a sequence $(a^{(j)})$ in the space E_R with

$$a_n^{(j)} = \begin{cases} a_n, & \text{if } n \leq j, \\ 0, & \text{otherwise.} \end{cases}$$

Then, since

$$\|a^{(j)} - a\|_\sigma = \sum_{n=1}^{\infty} \left| a_n^{(j)} - a_n \right| e^{-\sigma} = \sum_{n=j+1}^{\infty} \left| a_n^{(j)} - a_n \right| e^{-\sigma},$$

(3.4) shows that $a^{(j)} \to a$ with respect to the topology τ_σ.

On the other hand, concerning the convergence of this sequence in the topology ρ, we consider

$$\rho(a^{(j)}, a) = \sup_{n \geq j} |a_n|^{1/|\lambda_n|}.$$

The sequence $(\rho(a^{(j)}, a))$ need not tend to 0 as $j \to \infty$. For this claim, it is enough to give an example.

Indeed, if we take the sequence (a_n) defined as follows:

$$a_n = e^{-\sigma\lambda_n}, n = 1, 2, \ldots, \tag{3.5}$$

then $a = (a_n) \in E_R$. For this sequence (3.5), we have

$$|a_n|^{1/\lambda_n} = e^{-\sigma}, \ \forall n \geq 1.$$

Thus, the topology defined by metric ρ is strictly stronger than the Fréchet topology τ_σ and therefore cannot be locally convex. Indeed, if it were, we would have two topologies making E_R into a Fréchet space. These topologies would then be equivalent by the Banach homomorphism theorem: a contradiction. ■

4. Dual Spaces

Given two sequence spaces A and B, we denote by (A, B) the sequence space of "multipliers" from A to B,

$$(A, B) = \{u = (u_n); (u_n a_n) \in B, \ \forall (a_n) \in A\}.$$

A sequence space A is said to be normal [5] (or solid [1]) such that, whenever A contains (a_n), it also contains (b_n) with $|b_n| \leq |a_n|$ for $n = 1, 2, \ldots$. Equivalently, A is normal if $l^\infty \subset (A, A)$. For a sequence space A, there always exists the largest normal subspace denoted by $s(A)$ that is contained within it, and the smallest normal superspace denoted by $S(A)$ that contains it. More precisely, $s(A) = (l^\infty, A)$ and $S(A)$ is the intersection of all the normal spaces that contain A [1].

Various concepts of duality for sequence spaces are given in [2, 5]. Let D be a fixed sequence space. Then the D-dual of a sequence space A, denoted by A^D, is defined to be (A, D), the multipliers from A to D. The Köthe dual is obtained when $D = l^1$, and will be denoted by A^α (it is also denoted by A^K). The Abel dual is obtained when D is the space of Abel-summable sequences, that is, the space of sequences (d_n) for which

$$\lim_{r \to 1} \sum_{n=1}^{\infty} d_n r^n$$

exists. We denote the Abel dual of A by A^a. Note that when $d_n \geq 0$, the existence of this limit is equivalent to $\sum d_n < +\infty$. It is clear that $A^\alpha \subset A^a$. The inverse inclusion is true if space A is normal [1]. Spaces A^α and A^a were studied in [1,2,5,6].

In this section, we study some dual spaces of the space E_R.

We note that if the sequence (λ_n) satisfies condition (2.3), then

$$\sum_{n=1}^{\infty} r^{\lambda_n} < +\infty, \ \forall r \in (0, 1). \tag{4.1}$$

It is obvious that E_R is a normal space. Then $E_R^\alpha = E_R^a$.

Beside E_R^α, the Köthe dual of the space E_R, we introduce the following space:

$$E_R^\beta = \left\{(u_n); \sum_{n=1}^{\infty} u_n a_n \text{ converges}, \ \forall (a_n) \in E_R\right\}.$$

It is clear that $E_R^\alpha \subset E_R^\beta$.

Proposition 4.1. *If $(u_n) \in E_R^\beta$, then the following condition holds:*

$$\limsup_{n\to\infty} \frac{\log|u_n|}{\lambda_n} < -R. \tag{4.2}$$

Conversely, if the sequence (u_n) satisfies condition (4.2) and, in addition, the sequence (λ_n) satisfies condition (2.3), then $(u_n) \in E_R^\alpha$.

Proof. Let $(u_n) \in E_R^\beta$. Assume

$$\limsup_{n\to\infty} \frac{\log|u_n|}{\lambda_n} \geq -R,$$

the value of the left-hand side can be finite as well as $+\infty$. In any case, for a sequence $(\varepsilon_k) \downarrow 0$, there exists an increasing sequence (n_k) of positive numbers such that

$$\frac{\log|u_{n_k}|}{\lambda_{n_k}} \geq -R - \varepsilon_k, \forall k \geq 1,$$

which is equivalent to

$$\log\left(1/|u_{n_k}|\right) \leq (R+\varepsilon_k)\lambda_{n_k}.$$

Define a sequence (a_n) as follows:

$$a_n = \begin{cases} 1/|u_n|, & \text{if } n = n_k,\ k = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\limsup_{n\to\infty} \frac{\log|a_n|}{\lambda_n} \leq \limsup_{k\to\infty} \frac{\log|a_{n_k}|}{\lambda_{n_k}} \leq \limsup_{k\to\infty} \{(R+\varepsilon_k)\} = R,$$

which means that (a_n) is in E_R.

However, since $|a_n u_n| = 1$, for $n = n_k$ $(k = 1, 2, \ldots)$, it follows that $a_n u_n$ does not tend to 0 as $n \to \infty$. So the series $\sum_{n=1}^{\infty} a_n u_n$ does not converge, a contradiction.

Conversely, assume there exists a constant M such that (4.2) holds, i.e.,

$$\limsup_{n\to\infty} \frac{\log|u_n|}{\lambda_n} = M < -R,$$

and also condition (2.3) is satisfied.

Then for $\varepsilon > 0$ (satisfying $2\varepsilon < -R - M$), there exists N_1 such that $\forall n \geq N_1$

$$\frac{\log|u_n|}{\lambda_n} \leq M + \varepsilon,$$

or, equivalently,

$$|u_n| \leq e^{(M+\varepsilon)\lambda_n}.$$

On the other hand, for each $(a_n) \in E_R$, there exists N_2 such that $\forall n \geq N_2$

$$|a_n| \leq e^{(R+\varepsilon)\lambda_n}.$$

Hence, for all $n \geq \max\{N_1, N_2\}$,

$$|a_n u_n| \leq e^{(M+R+2\varepsilon)\lambda_n},$$

which implies that the series $\sum_{n=1}^{\infty} |a_n u_n|$ converges due to (4.1). ■

Corollary 4.2. *If (2.3) is satisfied, then* $(u_n) \in E_R^\beta$ *if and only if* $(u_n) \in E_R^\alpha$, *i.e.*, $E_R^\beta = E_R^\alpha$. *In this case, these spaces can be defined as follows:*

$$E_R^\beta = E_R^\alpha = \{(u_n) \text{ satisfies } (4.2)\}.$$

It is clear that $E_R \subset E_R^{\alpha\alpha}$. We shall prove that with condition (2.3), the inverse inclusion is true.

Proposition 4.3. *Suppose condition (2.3) holds. Then the space* E_R *is perfect, i.e.,* $E_R^{\alpha\alpha} = E_R$.

Proof. Suppose $(a_n) \notin E_R$. This means that

$$\limsup_{n\to\infty} \frac{\log|a_n|}{\lambda_n} > R;$$

the value of the left-hand side can be finite as well as $+\infty$. In any case, there exists $M > R > 0$ such that, for a sequence $(\varepsilon_k) \downarrow 0$, there exists an increasing sequence (n_k) of positive numbers such that

$$\frac{\log|a_{n_k}|}{\lambda_{n_k}} \geq M - \varepsilon_k, \ \forall k \geq 1,$$

which is equivalent to

$$\log\left(1/|a_{n_k}|\right) \leq (\varepsilon_k - M)\lambda_{n_k}.$$

Define a sequence (u_n) as follows:

$$u_n = \begin{cases} 1/|a_n|, & \text{if } n = n_k, \ k = 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\limsup_{n\to\infty} \frac{\log|u_n|}{\lambda_n} \leq \limsup_{k\to\infty} \frac{\log\left(1/|a_{n_k}|\right)}{\lambda_{n_k}} \leq \limsup_{k\to\infty} (\varepsilon_k - M) = -M < -R,$$

which means that (u_n) is in E_R^α.

However, since $|a_n u_n| = 1$, for $n = n_k$ $(k = 1, 2, \dots)$, the series $\sum_{n=1}^\infty a_n u_n$ does not converge. Hence, $(a_n) \notin E_R^{\alpha\alpha}$. The proof is complete. ∎

From now on, the sequence (λ_n) satisfying condition (2.3) is considered to be given.

Taking into account Proposition 4.1, we study a question about linear continuous functionals on the space E_R with the metric ρ. In a similar way, as was in [7], we can prove the following result.

Proposition 4.4. *Let* $a = (a_n) \in E_R$. *Then every linear continuous functional F from the dual space* E_R^* *has the form*

$$F(a) = \sum_{n=1}^\infty a_n u_n,$$

where (u_n) *satisfies condition (4.2), i.e.,* $(u_n) \in E_R^\alpha$.

As noted above, the Köthe dual of a sequence space is in fact the sequence space of multipliers from this space to the space l^1. A question arises: What about multipliers from E_R to l^p?

Theorem 4.5. *The following assertion holds:*

$$(E_R, l^p) = E_R^\alpha, \ \forall\, 0 < p \le \infty.$$

Proof. Let $(u_n) \in (E_R, l^p), 0 < p \le \infty$. Assume $(u_n) \notin E_R^\alpha$, which means that

$$\limsup_{n\to\infty} \frac{\log|u_n|}{\lambda_n} \ge -R;$$

the value of the left-hand side can be finite as well as $+\infty$. In any case, for a sequence $(\varepsilon_k) \downarrow 0$, there exists an increasing sequence (n_k) of positive numbers such that

$$\frac{\log|u_{n_k}|}{\lambda_{n_k}} \ge -R - \varepsilon_k, \forall k \ge 1,$$

which is equivalent to

$$\log\left(1/|u_{n_k}|\right) \le (R + \varepsilon_k)\lambda_{n_k}.$$

(1) In the case $0 < p < \infty$, consider a sequence

$$a_n = \begin{cases} 1/\left|u_{n_k}\right|, & \text{if } n = n_k,\ k = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\limsup_{n\to\infty} \frac{\log|a_n|}{\lambda_n} = \limsup_{k\to\infty} \frac{\log|a_{n_k}|}{\lambda_{n_k}}$$
$$= \limsup_{k\to\infty} \frac{\log\left(1/|u_{n_k}|\right)}{\lambda_{n_k}} \le \limsup_{k\to\infty} (R + \varepsilon_k) = R,$$

which means that $(a_k) \in E_R$.

However, since $|a_{n_k} u_{n_k}| = 1, \ (k = 1, 2, \ldots)$, we have $\sum_{n=1}^{\infty} |a_n u_n|^p = \infty$ and $(a_n u_n) \notin l^p$.

(2) In the case $p = \infty$, consider a sequence

$$a_n = \begin{cases} n_k/\left|u_{n_k}\right|, & \text{if } n = n_k,\ k = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\limsup_{n\to\infty} \frac{\log|a_n|}{\lambda_n} = \limsup_{k\to\infty} \frac{\log|a_{n_k}|}{\lambda_{n_k}} = \limsup_{k\to\infty} \frac{\log\left(n_k/|u_{n_k}|\right)}{\lambda_{n_k}}$$
$$\le \limsup_{k\to\infty} \frac{\log n_k}{\lambda_{n_k}} + \limsup_{k\to\infty} (R + \varepsilon_k) = R,$$

which means that $(a_k) \in E_R$.

However, since $|a_{n_k} u_{n_k}| = n_k \to +\infty$ as $k \to \infty$, it follows that $(a_n u_n) \notin l^\infty$.

So in both cases, we have that $(u_n) \notin (E_R, l^p)$, a contradiction. Thus, $(E_R, l^p) \subset E_R^\alpha$, $0 < p \le \infty$.

Conversely, let $(u_n) \in E_R^\alpha$. Then for some $M < -R$, there exists N_1 such that $\forall n \ge N_1$

$$|u_n| \le e^{M\lambda_n}.$$

Take an arbitrary $(a_n) \in E_R$. Then for $\varepsilon \in (0, -R - M)$, there exists N_2 such that $\forall n \ge N_2$

$$|a_n| \le e^{(R+\varepsilon)\lambda_n}.$$

Consequently, for all $n \ge N = \max\{N_1, N_2\}$, we have

$$|a_n u_n| \le e^{(M+R+\varepsilon)\lambda_n}.$$

(1) In the case $0 < p < \infty$, we have

$$\sum_{n=N}^{\infty} |a_n u_n|^p \le \sum_{n=N}^{\infty} e^{(M+R+\varepsilon)p\lambda_n} < \infty,$$

due to (4.1), as $M + R + \varepsilon < 0$, which means that $(a_n u_n) \in l^p$.

(2) In the case $p = \infty$, we have $|a_n u_n| \le e^{(M+R+\varepsilon)\lambda_n} \le 1$, $\forall n \ge N$, which shows that $(a_n u_n) \in l^\infty$.

Thus, in both cases $(u_n) \in (E_R, l^p)$ and $E_R^\alpha \subset (E_R, l^p)$, $0 < p \le \infty$. The proof is complete. ■

The next result concerns the sequence space of multipliers from l^p to E_R.

Theorem 4.6. *The following assertion holds*:

$$(l^p, E_R) = E_R, \forall\, 0 < p \le \infty.$$

Proof. Let $(u_n) \in (l^p, E_R)$, $0 < p \le \infty$. Assume $(u_n) \notin E_R$, which means that

$$\limsup_{n\to\infty} \frac{\log |u_n|}{\lambda_n} > R;$$

the value of the left-hand side can be finite as well as $+\infty$. In any case, there exists $M > R$ such that, for a sequence $(\varepsilon_k) \downarrow 0$, there exists an increasing sequence (n_k) of positive numbers such that

$$\frac{\log |u_{n_k}|}{\lambda_{n_k}} \ge M - \varepsilon_k, \ \forall k \ge 1,$$

which is equivalent to

$$\frac{1}{|u_{n_k}|} \le e^{\lambda_{n_k}(\varepsilon_k - M)}.$$

Define a sequence (ξ_n) as follows:

(1) In the case $0 < p < \infty$,

$$\xi_n = \begin{cases} e^{(M-\varepsilon_k-\nu)\lambda_{n_k}} / |u_{n_k}|, & \text{if } n = n_k,\ k = 1, 2, \dots, \\ 0, & \text{otherwise,} \end{cases}$$

where $\nu \in (0, M - R)$. Then we have

$$\sum_{n=1}^{\infty} |\xi_n|^p = \sum_{k=1}^{\infty} |\xi_{n_k}|^p = \sum_{k=1}^{\infty} \frac{e^{p\lambda_{n_k}(M-\varepsilon_k-\nu)}}{|u_{n_k}|^p} \le \sum_{k=1}^{\infty} e^{-p\nu\lambda_{n_k}} < +\infty,$$

due to (4.1), which shows that $(\xi_n) \in l^p$.

However,

$$\limsup_{n\to\infty} \frac{\log|\xi_n u_n|}{\lambda_n} = \limsup_{k\to\infty} \frac{\log|\xi_{n_k} u_{n_k}|}{\lambda_{n_k}} = \limsup_{k\to\infty} (M - \varepsilon_k - \nu) = M - \nu > R.$$

(2) In the case $p = \infty$,

$$\xi_n = \begin{cases} e^{(M-\varepsilon_k)\lambda_{n_k}} / |u_{n_k}|, & \text{if } n = n_k,\ k = 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have $|\xi_n| \le 1$, $\forall n \ge 1$, which implies that $(\xi_n) \in l^\infty$.
However,

$$\limsup_{n\to\infty} \frac{\log|\xi_n u_n|}{\lambda_n} = \limsup_{k\to\infty} \frac{\log|\xi_{n_k} u_{n_k}|}{\lambda_{n_k}} = \limsup_{k\to\infty} (M - \varepsilon_k) = M > R.$$

So in both cases, we have $(\xi_n u_n) \notin E_R$, a contradiction. Thus, $(l^p, E_R) \subset E_R$, $0 < p \le \infty$.

Conversely, let $(u_n) \in E_R$, which means that

$$\limsup_{n\to\infty} \frac{\log|u_n|}{\lambda_n} \le R.$$

Take an arbitrary $(\xi_n) \in l^p, 0 < p \le \infty$. In both cases, there exists a constant $M > 0$ such that $|\xi_n| \le M$, $\forall n \ge 1$. Then we have

$$\begin{aligned} \limsup_{n\to\infty} \frac{\log|\xi_n u_n|}{\lambda_n} &\le \limsup_{n\to\infty} \frac{\log|\xi_n|}{\lambda_n} + \limsup_{n\to\infty} \frac{\log|u_n|}{\lambda_n} \\ &\le \limsup_{n\to\infty} \frac{\log M}{\lambda_n} + \limsup_{n\to\infty} \frac{\log|u_n|}{\lambda_n} = \limsup_{n\to\infty} \frac{\log|u_n|}{\lambda_n} \le R, \end{aligned}$$

which shows that $(\xi_n u_n) \in E_R$.

Thus, $E_R \subset (l^p, E_R)$, $0 < p \le \infty$. The theorem is proved completely. ■

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