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Some Properties of Shape Metric

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Abstract. By slight modification of Borsuk's definition of fundamental metric ρ_F, we obtain in this note a definition of the shape metric ρ_S on the hyperspace 2^X of all non-empty uniformly bounded closed subsets of a complete metric space X.

Let us denote

$$FAEU(X) = \{A \in 2^X : A \text{ is an FAEU-space}\},$$
$$FANRU(X) = \{A \in 2^X : A \text{ is an FANRU-space}\}.$$

It is shown that $FAEU\ (X)$ is closed in $(2^X, \rho_S)$, but $FANRU\ (X)$ is not closed in $(2^X, \rho_S)$. The result gives partial answers to problems 8–9 and 9–9 posed by Borsuk [2].

1. Introduction

For the sake of convenience, throughout this note, "maps" mean uniformly continuous maps between metric spaces, and "neighborhoods" uniform neighborhoods.

Following Atsuji–Isbell [6], let us say that a subset A of a metric space X is uniformly bounded if and only if every uniformly continuous function on A is bounded.

Given a complete metric space X lying in an ANRU-space P, let 2^X denote the collection on all non-empty uniformly bounded closed subsets of X. For every A, B in 2^X, by $\rho_S(A, B)$, we denote the infimum of the set of all positive numbers ε satisfying the following condition.

There exist two fundamental sequences (see [7, 8])

$$\underline{f} = \{f_k, A, B\}_P \quad \text{and} \quad \underline{\hat{f}} = \{\hat{f}_k, B, A\}_P$$

such that there is a neighborhood (U, V) of the pair (A, B) in (P, P) such that for almost all k, $\rho(x, f_k(x)) \leq \varepsilon$ for every $x \in U$ and $\rho(x, \hat{f}_k(x)) \leq \varepsilon$ for every $x \in V$.

Remark. In [2], Borsuk introduced and studied the fundamental metric ρ_F on the hyperspace of all compact sets of a metric space X, and in [8], Nhu studied the weak metric ρ_W on the hyperspace of all uniformly bounded closed subsets of a complete metric space X. The definition of ρ_S is a slight modification of Borsuk's and Nhu's. By the same argument as that of Borsuk in [2], one shows the following:

Theorem 1.1. *ρ_S is a metric on 2^X and it does not depend on the ANRU-space P containing X isometrically.*

The metric ρ_S will be called the *shape metric* of 2^X. It is easy to see that $\rho_S(A, B) = \rho_F(A, B)$ for every compact sets $A, B \in 2^X$. The collection of all compact sets of X is closed in 2^X.

The aim of this note is to prove some properties of the shape metric ρ_S similar to that proved by Borsuk [2] for the fundamental metric ρ_F, and by Nhu [8] for the weak metric ρ_W. We prove first that the uniformity of 2^X depends only on the uniformity of X.

Theorem 1.2. *If X and Y are uniformly homeomorphic, so are 2^X and 2^Y.*

Proof. Consider X and Y as subsets of ANRU-spaces P and Q, respectively. Let $h : X \to Y$ be a uniform homeomorphism. Then there are a neighborhood (U_0, V_0) of (X, Y) in (P, Q) and maps $\varphi : U_0 \to Q$, $\psi : V_0 \to P$ such that $\varphi|X = h$ and $\psi|Y = h^{-1}$.

Let us define a map $h^\# : 2^X \to 2^Y$ by $h^\#(A) = h(A)$ for every $A \in 2^X$.

It is clear that $h^\#$ is a one-to-one map from 2^X onto 2^Y. Thus, by symmetry, it suffices to prove that $h^\#$ is uniformly continuous.

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\begin{aligned} &\rho(x, x') < \delta \Rightarrow \rho(\varphi(x), \varphi(x')) < \frac{\varepsilon}{2} \text{ for } x, x' \in X \\ &\rho(y, y') < \delta \Rightarrow \rho(\varphi(y), \varphi(y')) < \frac{\varepsilon}{2} \text{ for } y, y' \in Y. \end{aligned} \tag{1.1}$$

Let us show that

$$\rho_S(A, B) < \delta \Rightarrow \rho_S(h^\#(A), h^\#(B)) < \varepsilon.$$

By the definition of ρ_S, there exist two fundamental sequences

$$\underline{f} = \{f_k, A, B\}_P \quad \text{and} \quad \underline{\hat{f}} = \{\hat{f}_k, B, A\}_P$$

and a neighborhood (U, V) of (A, B) in (P, P) such that, for almost all k,

$$\begin{aligned} &\rho(x, f_k(x)) < \delta \text{ for every } x \in U \\ &\rho(x, \hat{f}_k(x)) < \delta \text{ for every } x \in V. \end{aligned} \tag{1.2}$$

Select a neighborhood $(\hat{U}, \hat{V}) \subset (\psi^{-1}(U), \psi^{-1}(V))$ of $(h(A), h(B))$ in (Q, Q) such that

$$\rho(y, \varphi\psi(y)) < \frac{\varepsilon}{2} \text{ for every } y \in \hat{U} \cup \hat{V}. \tag{1.3}$$

Setting $g_k = \varphi f_k \psi$, $\hat{g}_k = \varphi \hat{f}_k \psi$ for every $k = 1, 2 \ldots$, we obtain fundamental sequences

$$\underline{g} = \{g_k, h(A), h(B)\}_Q \quad \text{and} \quad \underline{\hat{g}} = \{\hat{g}_k, h(B), h(A)\}_Q.$$

If $y \in \hat{U}$, then $\psi(y) \in U$. Thus, by (1.2),

$$\rho(\psi(y), f_k\psi(y)) < \delta \text{ for almost all } k.$$

Hence, by (1.1),

$$\rho(\varphi\psi(y), \varphi f_k\psi(y)) < \frac{\varepsilon}{2} \text{ for almost all } k.$$

Consequently, from (1.3), we obtain

$$\rho(y, g_k(y)) < \varepsilon \text{ for almost all } k.$$

Similarly, we have

$$\rho(y, \hat{g}_k(y)) < \varepsilon \text{ for every } y \in \hat{V} \text{ and for almost all } k.$$

That means

$$\rho_S(h^\#(A), h^\#(B)) < \varepsilon.$$

The theorem is proved. ■

2. The Main Results

A metric space Y lying in an ANRU-space Q is called an *FAEU-space* [9] if and only if, for every subset A of a metric space X lying in an ANRU-space P and for every fundamental sequence $\underline{f} = \{f_k, A, Y\}_{P,Q}$, there exists a fundamental sequence $\underline{f'} = \{f'_k, X, Y\}_{P,Q}$ such that $f'_k(x) = f_k(x)$ for every $x \in A$ and $k = 1, 2, \ldots$.

In [9], it was shown that every FAEU-space is uniformly bounded. Let FAEU(X) denote the collection of all FAEU-spaces lying in a complete metric space X. We have the following:

Theorem 2.1. *FAEU(X) is closed in* 2^X.

We use the following lemma.

Lemma 2.2. *A metric space Y lying in an ANRU-space Q is an FAEU-space if and only if, for every neighborhood U of Y in Q, there exists a neighborhood U_0 of Y in Q such that, for every metric space X and for every map f from a subset A of X into U_0, there exists a map $\tilde{f}' : X \to U$ such that $\tilde{f}|A = f$.*

For the proof of Lemma 2.2, see [9].

Proof of Theorem 2.1. Consider X as a subset of the ANRU-space $P = l^\infty(X)$. Assume $A_0 \in FAEU(X)$ and let U be a neighborhood of A_0 in P. Put

$$\varepsilon = \frac{1}{8}\rho(A_0, P \setminus U). \tag{2.1}$$

Select an $A \in FAEU(X)$ such that $\rho_S(A, A_0) < \varepsilon$. It follows that there exist two fundamental sequences

$$\underline{f} = \{f_k, A, A_0\}_P \text{ and } \underline{\hat{f}} = \{\hat{f}_k, A_0, A\}_P$$

and a neighborhood (W, W_0) of (A, A_0) in (P, P) such that, for almost all k,

$$\begin{aligned} \rho(x, f_k(x)) &< \varepsilon \text{ for every } x \in W \\ \rho(x, \hat{f}_k(x)) &< \varepsilon \text{ for every } x \in W_0 . \end{aligned} \tag{2.2}$$

We may assume

$$\begin{aligned} W &\subset \{x \in P : \rho(x, A) \leq \varepsilon\} \\ W_0 &\subset \{x \in P : \rho(x, A_0) \leq \varepsilon\} . \end{aligned} \tag{2.3}$$

From (2.1)–(2.3), it follows that the set

$$V = \{x \in P : \rho(x, A_0) \leq 4\varepsilon\}$$

is a neighborhood of A in P and that

$$\hat{f}_k|_{W_0} \cong \mathrm{id}_{W_0} \text{ in } V \text{ for almost all } k. \tag{2.4}$$

Since $A \in FAEU(X)$, there exists a neighborhood V_0 of A in P such that every map with values in V_0 can be extended to a map with values in V. Let $U_0 \subset W_0$ be a neighborhood of A_0 in P such that

$$\hat{f}_k(U_0) \subset V_0 \text{ for almost all } k. \tag{2.5}$$

Select an index $k_0 \in N$ such that all the conditions (2.2), (2.4) and (2.5) are satisfied. Let φ be a map from a subset Y of a metric space Z into U_0. Then there exists a map $\psi : Z \to V$ such that $\psi|_Y = \widehat{f_{k_0}}\varphi$.

From (2.4), we obtain

$$\varphi \cong \widehat{f_{k_0}}\varphi \text{ in } V.$$

Thus, by the Homotopy Extension Lemma [7], there exists a map $\varphi' : Z \to U$ such that $\varphi'|_Y = \varphi$.

Then the theorem follows from Lemma 2.2.

Remark. In [2], Borsuk posed the following problem.

Problem. [2, 9.9] Let α be a hereditary shape property (that is, if $shX \geq shY$ and $X \in \alpha$, then $Y \in \alpha$). Is it true that, for every sequence $A_0, A_1, \ldots$ of compact sets lying in a metric space X, the two conditions

(1) $\lim_{n\to\infty} \rho_F(A_n, A_0) = 0$;
(2) $A_n \in \alpha$ for every $n = 1, 2, \ldots$

imply that $A_0 \in \alpha$?

Theorem 2.1 gives a positive answer to this problem for $\alpha = FAR$. The following example shows that the answer is negative if $\alpha = FANR$.

Example. Let us consider the sets

(1) $A_n = \{k^{-1}\}_{k=1}^{n}$ for every $n = 1, 2, \ldots$
(2) $A_0 = \{k^{-1}\}_{k=1}^{\infty} \cup \{0\}$.

It is easy to see that

(1) $\lim_{n\to\infty} \rho_F(A_n, A_0) = 0$
(2) $A_n \in FANR$ for every $n = 1, 2, \ldots$.

Let us show that

(3) $A_0 \notin FANR$.

Indeed, let U be a neighborhood of A_0 in R^1. We may assume U is of the form

$$U = \{x \in R^1 : \|x - A_0\| \leq \delta\}$$

for some $\delta > 0$. Then we have $[0, \delta] \subset U$.

Select an index $n_0 > \delta^{-1}$ and put

$$V = \left\{x \in R^1 : \|x - A_0\| \leq \frac{1}{4n_0(n_0+1)}\right\}.$$

It is easy to see that there is no continuous map r from U into V such that $r(x) = x$ for every $x \in A_0$. Thus, $A_0 \notin FANR$.

Since FAEU-spaces are the same as metric spaces with trivial shape [9], we obtain the following:

Corollary 2.3. *The set*

$$\{A \in 2^X : sh A \text{ is trivial}\}$$

is closed in 2^X.

Corollary 2.3 gives a partial answer to Problem 8–9 in [2] posed by Borsuk.

3. Some Closed Sets of 2^X

Definition 3.1. [3] *A metric space Y lying in an ANRU-space Q is called approximatively n-connected if and only if, for every neighborhood U of Y in Q, there exists a neighborhood U_0 of Y in Q such that every map from the n-sphere S^n into U_0 is homotopic in U to a constant map.*

Definition 3.2. [4] *A metric space Y lying in an ANRU-space Q is called movable if and only if, for every neighborhood U of Y in Q, there exists a neighborhood $U_0 \subset U$ of Y in Q such that, for every neighborhood V of Y in Q, there is a homotopy $\varphi : U_0 \times I \to U$ satisfying the condition*

$$\varphi(x, 0) = x, \quad \varphi(x, 1) \in V \text{ for every } x \in U_0.$$

Definition 3.3. [5] *Let X and Y be metric spaces lying in ANRU-spaces P and Q, respectively. Then Y is said to be quasi-dominated by X (notation $Y \leq X$) if and only if, for every neighborhood U of Y in Q, there exist a neighborhood V of Y in Q and two fundamental sequences*

$$\underline{f} = \{f_k, X, Y)_{P,Q} \text{ and } \underline{\hat{g}} = \{g_k, Y, X\}_{Q,P}$$

such that

$$f_k g_k|_V \cong \mathrm{id}_V \text{ in } U \text{ for almost all } k.$$

It is easy to see that in Definitions 3.1–3.3, the choice of ANRU-spaces $P \supset X$ and $Q \supset Y$ is immaterial.

Let us denote

$$\begin{aligned} \mathcal{C}_n(X) &= \{A \in 2^X : A \text{ is approximately } n\text{-connected}\} \\ \mathcal{M}(X) &= \{A \in 2^X : A \text{ is movable}\} \\ \mathcal{D}(X, B) &= \{A \in 2^X : A \leq B\}. \end{aligned}$$

Theorem 3.1. *$\mathcal{C}_n(X)$, $\mathcal{M}(X)$, $\mathcal{D}(X, B)$ are closed in 2^X.*

Proof. We prove the theorem only for $\mathcal{C}_n(X)$ because the proofs for $\mathcal{M}(X)$ and $\mathcal{D}(X, B)$ are similar (see [2]).

Consider X as a subset of the ANRU-space $P = l^\infty(X)$. Assume $A_0 \in \overline{\mathcal{C}_n(X)}$ and let U be a neighborhood of A_0 in P. Put

$$\varepsilon = \frac{1}{8}\rho(A_0, P \setminus U). \tag{3.1}$$

Select an $A \in \mathcal{C}_n(X)$ such that $\rho_S(A, A_0) < \varepsilon$. Then there exist two fundamental sequences $\underline{f} = \{f_k, A, A_0\}_P$ and $\underline{\hat{f}} = \{\hat{f}_k, A_0, A\}_P$ and a neighborhood (W, W_0) of (A, A_0) in (P, P) satisfying the following conditions for almost all k

$$\begin{aligned} &\rho(x, f_k(x)) < \varepsilon \text{ for every } x \in W \\ &\rho(x, \hat{f}_k(x)) < \varepsilon \text{ for every } x \in W_0. \end{aligned} \tag{3.2}$$

We may assume

$$W \subset \{x \in P : \rho(x, A) \leq \varepsilon\}, \quad W_0 \subset \{x \in P : \rho(x, A_0) \leq \varepsilon\}. \tag{3.3}$$

From (3.1)–(3.3), it follows that

$$\hat{f}_k|_{W_0} \cong \mathrm{id}_{W_0} \text{ in } U \text{ for almost all } k. \tag{3.4}$$

Since $A \in \mathcal{C}_n(X)$, there exists a neighborhood V_0 of A in P such that every map from the n-sphere S^n into V_0 is homotopic in U to a constant map. Let $U_0 \subset W_0$ be a neighborhood of A_0 in P such that

$$\hat{f}_k(U_0) \subset V_0 \text{ for almost all } k. \tag{3.5}$$

Select an index $k_0 \in N$ such that all conditions (3.2), (3.4) and (3.5) are satisfied and let $\varphi : S^n \to U_0$ be a map. Then $\widehat{f_{k_0}}\varphi$ is homotopic in U to a constant map. Hence, by (3.4), $\varphi \cong \widehat{f_{k_0}}\varphi$ in U. Thus, φ is homotopic in U to a constant map and the theorem is proved. ∎

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