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SLLN for Adapted Sequences

Fatima Ezzaki
Department of Mathematics
University of Sidi Mohammed Ben Abdellah, F.S.T Saiss V
Road Dimmouzer, B.P. 2202, Fès, Morocco

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Abstract. In the present paper, we examine the strong law of large numbers (SLLN) for integrable adapted sequences with values in a p-smooth (in Pisier's sense [11]) Banach space E. A general SLLN theorem for integrable adapted sequences is proved and a special form for *mils* [4] and *martingales difference* [10] is obtained. This leads to SLLN in Mosco convergence for convex weakly compact valued integrable adapted sequence including *super-martingales* and martingales difference.

1. Introduction

We shall suppose throughout this paper that $(\Omega, \mathcal{F}, P)$ is a complete probability space, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \ldots$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is an increasing sequence of sub-σ-algebras of $\mathcal{F}$, $\langle c_n \rangle$ is a decreasing sequence in $\mathbf{R}^+$ such that $\lim_{n\to\infty} c_n = 0$, E is a separable Banach space, $cc(E)$ (resp. $cwk(E)$) is the family of all nonempty closed (resp. weakly compact) convex subsets of E, and $\mathcal{E}$ is the Effros tribe on $cc(E)$. A closed convex valued measurable multifunction (alias *closed convex random set*) from Ω to $cc(E)$ is a $(\mathcal{F}, \mathcal{E})$-measurable mapping from Ω to $cc(E)$. A measurable multifunction X from Ω to $cc(E)$ is integrable (resp. integrably bounded) if the real function $d(0, X(.))$ (resp. $|X| : \omega \mapsto sup\{\|x\| : x \in X(\omega)\}$) is integrable.

If X is a $\mathcal{F}$-measurable and integrable closed convex random set and $\mathcal{B}$ is a sub-σ-algebra of $\mathcal{F}$, then there is a $\mathcal{B}$-measurable random closed convex random set G such that

$$\mathcal{S}^1_G(\mathcal{B}) = cl\{E^{\mathcal{B}}(f) : f \in \mathcal{S}^1_X(\mathcal{F})\},$$

where $\mathcal{S}^1_G(\mathcal{B})$ is the set of all $\mathcal{B}$-measurable and integrable selections of G and the closure being taken in L^1_E. Such a G is the multivalued conditional expectation of X relative to $\mathcal{B}$ and is denoted by $E^{\mathcal{B}}X$. Let $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ be the set of all $\mathcal{F}$-measurable integrably bounded multifunction X from Ω to $cwk(E)$. If the strong dual of E is separable, then by

[14], the conditional expectation of a random set $X \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ belongs to $\mathcal{L}^1_{cwk(E)}(\mathcal{B})$. A sequence $(X_n)_{n\geq 1}$ in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ is a *super-martingale* if $X_n \in \mathcal{L}^1_{cwk(E)}(\mathcal{F}_n)$ and $E^{\mathcal{F}_n} X_{n+1}(\omega) \subset X_n(\omega)$ for all $n \geq 1$ and almost surely (a.s. for short) $\omega \in \Omega$. Given a super-martingale $\langle X_n \rangle$ in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$, then by [6], there is a martingale $\langle f_n, \mathcal{F}_n \rangle$ in $L^1_E(\mathcal{F})$ such that, for each n, $f_n(\omega) \in X_n(\omega)$ a.s. Let us recall the following notions of p-smooth Banach space due to Pisier [11]. Let E be a Banach space and $p \in]1, 2]$. We say that E is a p-smooth space, if there exists an equivalent norm on E for which the *modulus of smoothness* ρ_E defined as

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x + ty\| + \|x - ty\| - 2) : \|x\| = \|y\| = 1\right\}$$

satisfies the following condition: There exists a constant k such that $\rho_E(t) \leq kt^p$ for each $t \in]0, \infty[$.

Also, we shall use the following limit notions. Let C_1, $C_2, \dots$ and C_∞ be closed convex subsets of E. We say that C_n *Mosco converges* to C_∞ if the two following inclusions are satisfied:

$$C_\infty \subset \text{s-li}\, C_n := \{x \in E : \|x - x_n\| \to 0;\ x_n \in C_n\}$$

$$\text{w-ls}\, C_n := \{x \in E : x_{n_k} \to x \text{ weakly};\ x_{n_k} \in C_{n_k}\} \subset C_\infty.$$

We shall suppose throughout this paper that $p \in]1, 2]$ and E is a p-smooth Banach space.

2. SLLN for Adapted Sequences

The following lemma is decisive in the statement of our main result.

Lemma 2.1. *Let F be a Banach space and $(g_n, \mathcal{F}_n)_{n\geq 1}$ a martingale in $L^1_F(\mathcal{F})$ such that*

(a) $\lim_{k\to\infty} c_k^p E|g_k|^p = 0$;

(b) $\sum_{k=1}^{\infty} (c_k^p - c_{k+1}^p) E|g_k|^p < \infty$.

Then $\lim_{n\to\infty} c_n g_n = 0$ *a.s.*

Proof. By the Chow inequality [3], for positive integrable submartingales, we have

$$\epsilon^p P\big[\max_{k\geq n} c_k^p |g_k|^p \geq \epsilon^p\big] \leq c_n^p\, E|g_n|^p + \sum_{k=n+1}^{\infty} c_k^p\, E(|g_k|^p - |g_{k-1}|^p).$$

By (a), it is enough to check that $\sum_{k=2}^{\infty} c_k^p\, E(|g_k|^p - |g_{k-1}|^p) < \infty$. But for all integers $m \geq 1$, we have

$$c_1^p\, E|g_1|^p + \sum_{k=2}^{m} c_k^p\, E(|g_k|^p - |g_{k-1}|^p) = \sum_{k=1}^{m-1} (c_k^p - c_{k+1}^p) E|g_k|^p + c_m^p\, E|g_m|^p.$$

Then by (a) and (b), we obtain

$$\begin{aligned}\lim_{m\to\infty}\Big(c_1^p E|g_1|^p+\sum_{k=2}^{m}c_k^p E(|g_k|^p-|g_{k-1}|^p)\Big)\\ &= c_1^p E|g_1|^p+\sum_{k=2}^{\infty}c_k^p E(|g_k|^p-|g_{k-1}|^p)\\ &= \sum_{k=1}^{\infty}(c_k^p-c_{k+1}^p)E|g_k|^p\\ &\leq \infty.\end{aligned}$$

Now, we are ready to prove the main result in this section.

Theorem 2.2. *Let* $(X_n)_{n\geq 1}$ *be a sequence in* $L_E^1(\mathcal{F})$. *Assume that the following conditions are satisfied:*

(i) $\left\langle S_n=\sum_{i=1}^{n}X_i,\mathcal{F}_n\right\rangle$ *is an adapted sequence;*

(ii) $\sum_{i=1}^{\infty}c_i^p\, E|X_i|^p<\infty$;

(iii) $\sum_{i=1}^{\infty}c_i(|E^{\mathcal{F}_{i-1}}X_i|)<\infty$ *a.s.*

Then $\lim_{n\to\infty}c_n\sum_{i=1}^{n}X_i(\omega)=0$ *a.s.*

Proof. By Doob's decomposition for adapted sequence [4, p. 144], there exist a martingale $(Y_n,\mathcal{F}_n)$ and a predictable sequence $(Z_n,\mathcal{F}_n)$ such that $\forall n\geq 1$, $S_n=Y_n+Z_n$, where $S_1=Y_1, Z_1=0, Y_{n+1}-Y_n=S_{n+1}-E^{\mathcal{F}_n}S_{n+1}$, and $Z_{n+1}-Z_n=E^{\mathcal{F}_n}S_{n+1}-S_n$.

First Step. Claim. $\lim_{n\to\infty}c_nY_n=0$ a.s. It is enough to check that conditions (a) and (b) of Lemma 2.1 are satisfied. Indeed, by Pisier's martingale inequality, there exists a positive constant B such that

$$\forall n\geq 1,\ \ E|Y_n|^p\leq B\sum_{i=1}^{n}E|Y_i-Y_{i-1}|^p \tag{2.1}$$

with $Y_0=0$. On the other hand,

$$\begin{aligned}E|Y_n-Y_{n-1}|^p &= E|(S_n-S_{n-1})-(Z_n-Z_{n-1})|^p\\ &\leq E(|X_n|+|Z_n-Z_{n-1}|)^p\\ &= E(|X_n|+|E^{\mathcal{F}_{n-1}}S_n-S_{n-1}|)^p\\ &\leq 2^p\,E|X_n|^p \end{aligned}\tag{2.2}$$

using Hölder's inequality. By (2.1) and (2.2), we have

$$E|Y_n|^p\leq 2^p\,B\sum_{i=1}^{n}E|X_i|^p. \tag{2.3}$$

By (ii), (2.3) and Kronecker's lemma, we obtain (a). Moreover, by (2.3), we have

$$\begin{aligned}\sum_{n=1}^{\infty}(c_n^p - c_{n+1}^p)E|Y_n|^p &\leq 2^p B\sum_{n=1}^{\infty}(c_n^p - c_{n+1}^p)E\Big(\sum_{i=1}^{n}|X_i|^p\Big)\\ &= 2^p B\sum_{i=1}^{\infty}E|X_i|^p\sum_{n=i}^{\infty}(c_n^p - c_{n+1}^p)\\ &= 2^p B\sum_{i=1}^{\infty}c_i^p\, E|X_i|^p < \infty.\end{aligned}$$

Then (b) is satisfied. Hence, Lemma 2.1 gives the desired conclusion.

Second Step. Claim. $\lim_{n\to\infty} c_n Z_n = 0$ a.s. We have

$$\begin{aligned}|c_n Z_n| &= \Big|c_n\sum_{i=2}^{n}(Z_i - Z_{i-1})\Big|\\ &= \Big|c_n\sum_{i=2}^{n}(E^{\mathcal{F}_{i-1}}S_i - S_{i-1})\Big|\\ &\leq c_n\sum_{i=1}^{n}|E^{\mathcal{F}_{i-1}}X_i|.\end{aligned}$$

Applying again Kronecker's lemma in connection with (iii) yields $\lim_{n\to\infty} c_n Z_n = 0$ a.s., thus proving the theorem. ■

A special form of Theorem 2.2 is the following result of SLLN for mils.

Corollary. *Let* $\langle S_n = \sum_{i=1}^{n} X_i, \mathcal{F}_n\rangle$ *be an integrable mil such that*

$$\sum_{i=1}^{\infty}\frac{1}{i^p}E|X_i|^p < \infty.$$

Then $\lim_{n\to\infty}\frac{1}{n}S_n = 0$ *a.s.*

Proof. By applying Doob's decomposition to $\langle S_n = \sum_{i=1}^{n} X_i, \mathcal{F}_n\rangle$, there exist a martingale $\langle Y_n, \mathcal{F}_n\rangle$ and a predictable sequence $\langle Z_n, \mathcal{F}_n\rangle$ such that $\forall n$, $S_n = Y_n + Z_n$ and $Z_{n+1} - Z_n = E^{\mathcal{F}_n}S_{n+1} - S_n$. Set $c_i = \frac{1}{i}$, then $(S_n, \mathcal{F}_n)_{n\geq 1}$ satisfies conditions (i) and (ii) of Theorem 2.2. By repeating the first step of the proof in Theorem 2.2, we have $\lim_{n\to\infty}\frac{1}{n}Y_n = 0$ a.s. Now, we need to show that $\lim_{n\to\infty}\frac{1}{n}Z_n = 0$ a.s. Since $\langle S_n, \mathcal{F}_n\rangle$ is a mil, $\lim_{m\to\infty}\sup_{n\geq m}|E^{\mathcal{F}_m}S_n - S_m| = 0$ a.s., then

$$\lim_{n\to\infty}|Z_n - Z_{n-1}| = 0 \tag{2.4}$$

and since $\frac{1}{n} Z_n = \frac{1}{n}\sum_{i=1}^{n}(Z_i - Z_{i-1})$ with $Z_0 = 0$, so by (2.4), we deduce that $\lim_{n\to\infty} \frac{1}{n} Z_n = 0$ a.s. ■

3. SLLN for Convex Weakly Compact Adapted Sequences

The following is a multivalued version of Theorem 2.2.

Proposition 3.1. *Let X_1 be a random set in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$, $(X_i)_{i\geq 2}$ a sequence in $L^1_E(\mathcal{F})$ and $\langle S_n = X_1 + X_2 + \cdots + X_n, \mathcal{F}_n\rangle$ an adapted sequence. Assume the two following conditions are satisfied:*

(i) $\sum_{i=1}^{\infty} c_i^p\, E|X_i|^p < \infty$;

(ii) $\sum_{i=1}^{\infty} c_i |E^{\mathcal{F}_{i-1}} X_i| < \infty$ *a.s.*

Then $0 \in$ s-li $c_n S_n$ a.s.

Proof. Since X_1 is $\mathcal{F}_1$-measurable, there exists an integrable selection $h_1 \in \mathcal{S}^1_E(\mathcal{F}_1)$. So $\langle f_n = h_1 + X_2 + \cdots + X_n, \mathcal{F}_n\rangle$ is an adapted sequence selection of S_n. By (i), (ii) and Theorem 2.2, we deduce that $\lim_{n\to\infty} c_n f_n = 0$, then $0 \in$ s-li $c_n S_n$ a.s. ■

The following proposition ensures the existence of convex weakly compact valued adapted sequences having the form given in Proposition 3.1.

Proposition 3.2. *Let F be a Banach space with strongly separable dual F^*. If (X_n) is a sequence in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$, such that $\langle \sum_{i=1}^{n} X_i, \mathcal{F}_n\rangle$ is a super-martingale, then S_n has the form*

$$S_n = X_1 + f_2 + \cdots + f_n,$$

where $f_i \in L^1_F(\mathcal{F})$ $(i \geq 2)$.

Proof. Since $\langle S_n, \mathcal{F}_n\rangle$ is a super-martingale, it is easy to check that

$$\int_\Omega X_{n+1}(\omega)P(d\omega) = \{0\}$$

for all $n \geq 1$. From [7], there exists $f_{n+1} \in L^1_F(\mathcal{F})$ such that $X_{n+1}(\omega) = \{f_{n+1}(\omega)\}$ a.s. Then $\forall n \geq 1$

$$S_n(\omega) := \sum_{i=1}^{n} X_i(\omega) = X_1(\omega) + f_2(\omega) + \cdots + f_n(\omega) \text{ a.s.}$$ ■

The following is a consequence of Lemma 2.1 and Proposition 3.2.

Proposition 3.3. *Assume E is separable. Let (X_n) be a sequence in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ satisfying the following two conditions:*

(i) $\langle \sum_{i=1}^{n} X_i, \mathcal{F}_n \rangle$ *is a super-martingale;*

(ii) $\sum_{i=1}^{\infty} c_i^p E|X_i|^p < \infty$.

Then

(a) $0 \in \text{s-li}\, c_n \sum_{i=1}^{n} X_i(\omega)$ *a.s.;*

(b) *if* $\langle \sum_{i=1}^{n} X_i, \mathcal{F}_n \rangle$ *is a martingale, then for a.s.* $\omega \in \Omega$,

$$\text{M-}\lim_{n\to\infty} c_n \sum_{i=1}^{n} X_i(\omega) = \{0\}.$$

Proof Claim 1. $0 \in \text{s-li}\, c_n \sum_{i=1}^{n} X_i$ a.s.

First Proof. By (i) and Proposition 3.2, there exists $f_i \in L^1_E(\mathcal{F})$ $(i \geq 2)$ such that $\forall n \geq 1$,

$$S_n(\omega) := \sum_{i=1}^{n} X_i(\omega) = X_1(\omega) + f_2(\omega) + \cdots + f_n(\omega) \text{ a.s.}$$

Since $\langle S_n \rangle$ is a super-martingale in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$, by [5], there exists an integrable martingale selection $\langle g_n \rangle$ for $\langle S_n \rangle$. Consequently, for each $n \geq 1$, there exists an integrable selection h_1^n of X_1 such that a.s.

$$g_n = h_1^n + f_2 + \cdots + f_n. \tag{3.1}$$

Since $(g_n, \mathcal{F}_n)_{n\geq 1}$ is a martingale, it is enough to check that conditions (a) and (b) of Lemma 2.1 are satisfied. Indeed by (3.1), we have $h_1^n = h_1^{n+1} + E^{\mathcal{F}_n} f_{n+1}$, then $g_{n+1} - g_n = f_{n+1} - E^{\mathcal{F}_n} f_{n+1}$ where $g_0 = 0$, $f_1 = g_1$. On the other hand, by Pisier's martingale inequality, there exists a positive constant B such that $\forall n \geq 1$,

$$\begin{aligned} E|g_n|^p &\leq B \sum_{i=1}^{n} E|g_i - g_{i-1}|^p \\ &= B \sum_{i=1}^{n} E|f_i - E^{\mathcal{F}_{i-1}} f_i|^p \\ &\leq 2^p B \sum_{i=1}^{n} E|f_i|^p \end{aligned} \tag{3.2}$$

by using Hölder's inequality. By (3.2), we have

$$E|g_n|^p \leq 2^p B \sum_{i=1}^{n} E|f_i|^p. \tag{3.3}$$

By (ii), (3.3) and Kronecker's lemma, we obtain Lemma 2.1(a). Moreover, by (3.3), we have

$$\begin{aligned}\sum_{n=1}^{\infty}(c_n^p - c_{n+1}^p)E|g_n|^p &\leq 2^p B \sum_{n=1}^{\infty}(c_n^p - c_{n+1}^p)E\Big(\sum_{i=1}^{\infty}|f_i|^p\Big)\\ &\leq 2^p B \sum_{i=1}^{\infty} E|X_i|^p \sum_{n=i}^{\infty}(c_n^p - c_{n+1}^p)\\ &= 2^p B \sum_{i=1}^{\infty} c_i^p \, E|X_i|^p < \infty.\end{aligned}$$

Then Lemma 2.1(b) is satisfied. Hence, Lemma 2.1 gives the desired conclusion.

Second Proof. (Suggested by C. Hess) By (i), one can check that $\int_B f_{n+1}\, P(d\omega) = \{0\}$, $\forall B \in \mathcal{F}_n$ so that, for all n, $E^{\mathcal{F}_n}(f_{n+1}) = 0$ a.s. Set $\widetilde{S}_n = \widetilde{f}_1 + f_2 + \cdots + f_n$ (where $\widetilde{f}_1 = 0$), then $(\widetilde{S}_n, \mathcal{F}_n)_{n\geq 1}$ is an adapted sequence, and by (ii), $\sum_{i=2}^{\infty} c_i^p\, E|f_i|^p < \infty$. So by Proposition 3.1, we obtain $\lim_{n\to\infty} c_n(f_2 + f_3 + \cdots + f_n) = 0$ a.s., and since $c_n|X_1| \to 0$ a.s., (a) is satisfied. It remains to prove:

Claim 2. w-ls $c_n \sum_{i=1}^{n} X_i(\omega) = \{0\}$ a.s. Let $(e_k^*)_{k\geq 1}$ be a dense sequence in E^*.

Applying Theorem 2.2 to real valued martingales $\langle \delta^*(e_k^*, S_n)\rangle$ yields

$$\lim_{n\to\infty} c_n \delta^*(e_k^*, S_n(\omega)) = 0 \ \text{ a.s.}$$

for all $k \geq 1$. Hence, Claim 2 follows. ■

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