Short Communication

Note on the Kolmogorov–Stein Inequality*

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A. N. Kolmogorov has given the following result [4]: Let \( f(x), f'(x), \ldots, f^{(n)}(x) \) be continuous and bounded on \( \mathbb{R} \). Then

\[
\|f^{(k)}\|_\infty \leq C_{k,n} \|f\|_\infty^{n-k} \|f^{(n)}\|_\infty^k,
\]

where \( 0 < k < n \), \( C_{k,n} = K_n^{n-k}/K_n^{(n-k)} \), \( K_i = \frac{4}{\pi} \sum_{j=0}^{\infty} (-1)^j/(2j+1)^{i+1} \) for even \( i \), while \( K_i = 4 \sum_{j=0}^{\infty} 1/(2j+1)^{i+1} \) for odd \( i \). Moreover, the constants are best possible.

This result has been extended by E. M. Stein to \( L_p \)-norm [7] and by Ha Huy Bang to any Orlicz norm [1]. The Kolmogorov–Stein inequality and its variants are an interest for many mathematicians and have various applications (see, for example, [2, 8] and their references).

In this paper, modifying the methods of [1, 7], we prove this inequality for another norm generated by concave functions. Note that the Orlicz norm is generated by convex functions and here we must overcome some essential difficulties because of the difference between convex and concave functions.

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Let $L$ denote the family of all non-zero concave functions $\Phi(t) : [0, \infty) \to [0, \infty]$, which are non-decreasing and satisfy $\Phi(0) = 0$. Denote by $N_\Phi = N_\Phi(\mathbb{R})$ the space of measurable functions $f(x)$ such that $\|f\|_{N_\Phi} = \int_0^\infty \Phi(\lambda_f(y)) dy < \infty$, where $\lambda_f(y) = \text{mes}\{x : |f(x)| > y\}$, $(y \geq 0)$, and by $M_\Phi = M_\Phi(\mathbb{R})$, the space of measurable functions $g(x)$ such that

$$\|g\|_{M_\Phi} = \sup \left\{ \frac{1}{\Phi(\text{mes} \Delta)} \int_{\Delta} |g(x)| dx : \Delta \subset \mathbb{R}, \ 0 < \text{mes} \Delta < \infty \right\} < \infty.$$ 

Then $N_\Phi$ and $M_\Phi$ are Banach spaces [5, 6].

The main result of this paper is the following:

**Theorem 1.** Let $\Phi \in L$, $f(x)$ and its generalized derivative $f^{(n)}(x)$ be in $N_\Phi$. Then $f^{(k)}(x) \in N_\Phi$ for all $0 < k < n$ and

$$\|f^{(k)}\|_{N_\Phi} \leq C_{k,n} \|f\|_{N_\Phi}^{n-k} \|f^{(n)}\|_{N_\Phi}^{k}. \quad (1)$$

**Proof.** The sketch of proof is as follows. We begin to prove (1) with the assumption that $f^{(k)}(x) \in N_\Phi$, $0 \leq k \leq n$.

By Theorem 4.3 in [6] and [9, p. 113], we have

$$\|f^{(k)}\|_{N_\Phi} = \sup_{\|g\|_{M_\Phi} = 1} \left| \int_{-\infty}^\infty f^{(k)}(x) g(x) dx \right|. \quad (2)$$

Let $\varepsilon > 0$. We choose a function $h(x) \in M_\Phi$ such that $\|h\|_{M_\Phi} = \varepsilon$ and

$$\left| \int_{-\infty}^\infty f^{(k)}(x) h(x) dx \right| \geq \|f^{(k)}\|_{N_\Phi} - \varepsilon. \quad (3)$$

Put

$$F(x) = \int_{-\infty}^x f(x + y) h(y) dy.$$ 

By Theorem 4.4 in [6], we obtain $F(x) \in L_\infty(\mathbb{R})$. Arguing as in [1], we have

$$F^{(r)}(x) = \int_{-\infty}^x f^{(r)}(x + y) h(y) dy, \quad 0 \leq r \leq n \quad (4)$$

in the distribution sense. It is easy to check that $|F^{(r)}(x)| \leq \|f^{(r)}\|_{N_\Phi}, \forall x \in \mathbb{R}.$

Now, we prove the continuity of $F^{(r)}(x)$ on $\mathbb{R}$ $(0 \leq r \leq n)$. We show this for $r = 0$ by contradiction: Assume that for some $\varepsilon > 0$, point $x^0$ and subsequence $|t_k| \to 0$, $\left| \int_{-\infty}^\infty \left(f(x^0 + t_k + y) - f(x^0 + y)\right) h(y) dy \right| \geq \varepsilon, \ k \geq 1. \quad (5)$

Since $f \in N_\Phi$, we easily obtain $f \in L_{1,\text{loc}}(\mathbb{R})$. Then for any $m = 1, 2, \ldots, f(t_k + y) \to f(y)$ in $L_1(-m, m)$. Therefore, there exists a subsequence, denoted again by $\{t_k\}$, such that $f(t_k + y) \to f(y)$ a.e. in $(-m, m)$. Therefore, there exists a subsequence (for simplicity of notation, we assume it is coincident with $\{t_k\}$) such that $f(x^0 + t_k + y) \to f(x^0 + y)$ a.e. in $(-\infty, \infty)$. 


On the other hand, \( \{f(x_0 + t_\ast + y)\} \) is bounded in \( N_\Phi \). So \( \{f(x_0 + t_\ast + y)\} \) is a weak precompact sequence. Therefore, there exist a subsequence denoted by \( \{f(x_0 + t_\ast + y)\} \) and a function \( f_\ast(y) \in N_\Phi \) such that

\[
\int_{-\infty}^{\infty} f(x_0 + t_\ast + y) v(y) dy \rightarrow \int_{-\infty}^{\infty} f_\ast(y) v(y) d(y), \ \forall \ v(y) \in M_\Phi. \tag{6}
\]

Let \( u(x) \) be an arbitrary function in \( C_0^\infty(\mathbb{R}) \), then \( u(x) \in M_\Phi \). It follows from \( f(x_0 + t_\ast + y) \rightarrow f(x_0 + y) \) a.e. that

\[
\int_{-\infty}^{\infty} f(x_0 + t_\ast + y) u(y) dy \rightarrow \int_{-\infty}^{\infty} f(x_0 + y) u(y) dy, \ \forall \ u \in C_0^\infty(\mathbb{R}). \tag{7}
\]

Combining (6) and (7) and \([3, \text{p. 15}]\), we obtain

\[
\int_{-\infty}^{\infty} f(x_0 + t_\ast + y) h(y) dy \rightarrow \int_{-\infty}^{\infty} f(x_0 + y) h(y) dy
\]

which contradicts (5). The cases \( 1 \leq r \leq n \) are proved similarly.

Therefore, it follows from the Kolmogorov inequality and (3) and (4) that

\[
\left( \| f^{(k)} \|_{N_\Phi} - \varepsilon \right)^n \leq |F^{(k)}(0)|^n \leq C_{k,n} \| F \|_{n-k}^n \| F^{(n)} \|_\infty^k. \tag{8}
\]

Because of \( \| F \|_\infty \leq \| f \|_{N_\Phi}, \| F^{(n)} \|_\infty \leq \| f^{(n)} \|_{N_\Phi} \) and by letting \( \varepsilon \rightarrow 0 \), we have (1).

It remains to show that \( f^{(k)} \in N_\Phi, 0 < k < n \) if \( f, f^{(n)} \in N_\Phi \).

Let \( \psi_\lambda(x) \in C_0^\infty(\mathbb{R}) \), \( \psi_\lambda(x) \geq 0, \psi_\lambda(x) = 0 \) for \( |x| \geq \lambda \) and \( \int \psi_\lambda(x) dx = 1 \). We put \( f_\lambda = f * \psi_\lambda \). Then \( f_\lambda \in C^\infty(\mathbb{R}) \). It is easy to check that \( f^{(k)}_\lambda = f * \psi^{(k)}_\lambda \in N_\Phi, k \geq 0 \) and \( f^{(n)}_\lambda = f^{(n)} * \psi_\lambda \). Therefore,

\[
\| f^{(k)}_\lambda \|_{N_\Phi}^2 \leq C_{k,n} \| f_\lambda \|_{N_\Phi}^{n-k} \| f^{(n)}_\lambda \|_{N_\Phi}^k, 0 < k < n.
\]

Since \( \| f_\lambda \|_{N_\Phi} \leq \| f \|_{N_\Phi}, \| f^{(n)}_\lambda \|_{N_\Phi} \leq \| f^{(n)} \|_{N_\Phi} \), we have that, for any \( 0 \leq k \leq n \), the sequence \( \{f^{(k)}_\lambda\} \) is bounded in \( N_\Phi \).

By an argument similar to the previous one, we obtain \( f_\lambda \rightarrow f \). Therefore, it follows that, for any \( \varphi \in C_0^\infty(\mathbb{R}) \),

\[
\left( f^{(k)}_\lambda(x), \varphi(x) \right) \rightarrow \left( f^{(k)}(x), \varphi(x) \right).
\]

So \( f^{(k)} \in N_\Phi \) for all \( 0 < k < n \) if \( f, f^{(n)} \in N_\Phi \). The proof is complete.

\[ \blacksquare \]

Remark. For periodic functions, we have
Theorem 2. Let $\Phi(t) \in \mathcal{L}$, $f(x)$ and its generalized derivative $f^{(n)}(x)$ be in $N_{\Phi}(T)$. Then $f^{(k)}(x) \in N_{\Phi}(T)$ for all $0 < k < n$ and

$$
\| f^{(k)} \|_{N_{\Phi}(T)} \leq C_{k,n} \| f \|_{N_{\Phi}(T)}^{n-k} \| f^{(n)} \|_{N_{\Phi}(T)}^{k},
$$

where $T$ is the torus and $\| \cdot \|_{N_{\Phi}(T)}$ is the corresponding norm.

Applying the obtained results, we can obtain imbedding theorems for spaces of infinite order. We give here, for example, one result.

Let $1 \leq q \leq \infty$, $\Phi \in \mathcal{L}$, and $a = \{a_k\}_{k \in P}$ be a sequence of non-negative real numbers, which contains an infinite subsequence of positive numbers, where $P$ is the set of non-negative integers. We denote by $W_{a,\Phi,q}^{\infty}$ the space of functions $f$ on the real line $\mathbb{R}$ whose following seminorms are finite:

$$
\| f \|_{a,q} = \left\{ \sum_{n \in P} (a_n \| f^{(n)} \|_{N_{\Phi}})^q \right\}^{1/q} (q < \infty),
$$

$$
\| f \|_{a,\infty} = \sup_{n \in P} \{a_n \| f^{(n)} \|_{N_{\Phi}}\} (q = \infty).
$$

The spaces $W_{a,\Phi,q}^{\infty}$ are called Sobolev spaces of infinite order. The space $W_{b,\Phi,q}^{\infty}$ is defined similarly.

Theorem 3. If the following imbedding holds:

$$
W_{a,\Phi,q}^{\infty} \hookrightarrow W_{b,\Phi,q}^{\infty},
$$

then there exists a constant $M$ such that

$$
F_{b,q}(t) \leq M F_{a,q}(t) \quad \forall t \geq 0,
$$

where

$$
F_{a,q}(t) = \begin{cases} 
\sum_{n \in P} a_n^q t^n & (q < \infty) \\
\sup_{n \in P} \{a_n t^n\} & (q = \infty).
\end{cases}
$$

References