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Short Communication

Spherical Classes and the Homology of the Steenrod Algebra*

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Let $P_k = \mathbf{F}_2[x_1, \ldots, x_k]$ be the polynomial algebra on k generators $x_1, \ldots, x_k$, each of dimension 1. Here, $\mathbf{F}_2$ denotes the field of two elements. Let the general linear group $GL_k = GL(k, \mathbf{F}_2)$ and the mod 2 Steenrod algebra $\mathcal{A}$ act on P_k in the usual way.

The Dickson algebra of k variables, D_k, is the algebra of invariants

$$D_k := \mathbf{F}_2[x_1, \ldots, x_k]^{GL_k}.$$

As the action of $\mathcal{A}$ and that of GL_k on P_k commute with each other, D_k is an algebra over $\mathcal{A}$. We are interested in the Lannes–Zarati homomorphism

$$\varphi_k : \mathrm{Ext}_{\mathcal{A}}^{k,k+i}(\mathbf{F}_2, \mathbf{F}_2) \to (\mathbf{F}_2 \underset{\mathcal{A}}{\otimes} D_k)_i^*,$$

which is compatible with the Hurewicz one

$$H : \pi_*^s(S^0) \cong \pi_*(Q_0S^0) \to H_*(Q_0S^0),$$

(see [9, 10, p. 46]). Here and throughout the paper, the coefficient ring for homology and cohomology is always $\mathbf{F}_2$. The definition of φ_k will be recalled later.

The classical conjecture on spherical classes reads as follows.

Conjecture 1. (Conjecture on Spherical Classes) *There are no spherical classes in Q_0S^0 except the elements of Hopf invariant one and those of Kervaire invariant one.*

(See [4, 16, 17] for a discussion.)

The Hopf invariant one and the Kervaire invariant one elements are respectively represented by certain permanent cycles in $\mathrm{Ext}_{\mathcal{A}}^{1,*}(\mathbf{F}_2, \mathbf{F}_2)$ and $\mathrm{Ext}_{\mathcal{A}}^{2,*}(\mathbf{F}_2, \mathbf{F}_2)$, on which φ_1 and φ_2 are non-zero (see [1, 3, 10].)

Conjecture 1 is a consequence of the following:

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Conjecture 2. *$\varphi_k = 0$ in any positive stem i for $k > 2$.*

(See [6–8] for a discussion.)

To state our main result, we need to summarize Singer's invariant-theoretic description of the Lambda algebra [14]. According to Dickson [5], one has

$$D_k \cong \mathbf{F}_2[Q_{k,k-1}, \ldots, Q_{k,0}],$$

where $Q_{k,i}$ denotes the Dickson invariant of dimension $2^k - 2^i$. Singer sets $\Gamma_k = D_k[Q_{k,0}^{-1}]$, the localization of D_k given by inverting $Q_{k,0}$, and defines $\Gamma_k^\wedge$ to be a certain "not too large" submodule of Γ_k. He also equips $\Gamma^\wedge = \oplus_k \Gamma_k^\wedge$ with a differential $\partial : \Gamma_k^\wedge \to \Gamma_{k-1}^\wedge$ and a coproduct. Then he shows that the differential coalgebra $\Gamma^\wedge$ is dual to the Lambda algebra of [2]. Thus, $H_k(\Gamma^\wedge) \cong \mathrm{Tor}_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$.

The main result of this paper is the following theorem, which has been conjectured in our paper [7, Conjecture 5.3].

Theorem 1. *The inclusion $D_k \subset \Gamma_k^\wedge$ is a chain-level representation of the Lannes–Zarati dual homomorphism*

$$\varphi_k^* : (\mathbf{F}_2 \underset{\mathcal{A}}{\otimes} D_k)_i \to \mathrm{Tor}_{k,k+i}^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2).$$

An immediate consequence of this theorem is the equivalence between Conjecture 2 and the following:

Conjecture 3. *If $q \in D_k^+$, then $[q] = 0$ in $\mathrm{Tor}_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$ for $k > 2$.*

This has been established for $k = 3$ in Theorem 4.8 in [7], while Conjecture 2 has been proved for $k = 3$ in Corollary 3.5 in [6]. From the viewpoint of this conjecture, it seems that Singer's model of the dual of the Lambda algebra, $\Gamma^\wedge$, is somehow more natural than the Lambda algebra itself.

Now, we recall the definition of φ_k^* after [10].

Let $P_1 = \mathbf{F}_2[x]$ with $|x| = 1$. Let $\hat{P} \subset \mathbf{F}_2[x, x^{-1}]$ be the submodule spanned by all powers x^i with $i \geq -1$. The canonical $\mathcal{A}$-action on P_1 is extended to an $\mathcal{A}$-action on $\mathbf{F}_2[x, x^{-1}]$. Then $\hat{P}$ is an $\mathcal{A}$-submodule of $\mathbf{F}_2[x, x^{-1}]$. One has a short-exact sequence of $\mathcal{A}$-modules

$$0 \to P_1 \xrightarrow{\iota} \hat{P} \xrightarrow{\pi} \Sigma^{-1}\mathbf{F}_2 \to 0,$$

where ι is the inclusion and π is given by $\pi(x^i) = 0$ if $i \neq -1$ and $\pi(x^{-1}) = 1$. Let e_1 be the corresponding element in $\mathrm{Ext}_{\mathcal{A}}^1(\Sigma^{-1}\mathbf{F}_2, P_1)$.

Definition 1. [15]

(i) $e_k = \underbrace{e_1 \otimes \cdots \otimes e_1}_{k \text{ times}} \in \mathrm{Ext}_{\mathcal{A}}^k(\Sigma^{-k}\mathbf{F}_2, P_k)$.

(ii) *$e_k(M) = e_k \otimes M \in \mathrm{Ext}_{\mathcal{A}}^k(\Sigma^{-k}M, P_k \otimes M)$, for M a left $\mathcal{A}$-module. Here, M also means the identity map of M.*

Following [10], the destabilization of M is defined by $\mathcal{D}M = M/EM$, where $EM := \mathrm{Span}\{Sq^i x | i > \deg x, x \in M\}$. They show that the functor associating M to $\mathcal{D}M$ is a right exact functor. Then they define $\mathcal{D}_k$ to be the kth left derived functor of $\mathcal{D}$. That means $\mathcal{D}_k(M) = H_k(\mathcal{D}F_*(M))$, where $F_*(M)$ is an $\mathcal{A}$-free (or $\mathcal{A}$-projective) resolution of M.

The cap-product with $e_k(M)$ gives rise to the homomorphism

$$e_k(M) : \mathcal{D}_k(\Sigma^{-k}M) \to \mathcal{D}_0(P_k \otimes M) \equiv P_k \otimes M$$
$$e_k(M)(z) = e_k(M) \cap z\,.$$

Theorem 2. [10] *Let $D_k \subset P_k$ be the Dickson algebra of k variables. Then $\alpha_k := e_k(\Sigma\mathbf{F}_2) : \mathcal{D}_k(\Sigma^{1-k}\mathbf{F}_2) \to \Sigma D_k$ is an isomorphism of internal degree 0.*

This theorem will explicitly be formulated in Proposition 2 below.

By definition of the functor $\mathcal{D}$, one has a natural homomorphism : $\mathcal{D}(M) \to \mathbf{F}_2 \underset{\mathcal{A}}{\otimes} M$. Then it induces a homomorphism $i_k : \mathbf{F}_2 \underset{\mathcal{A}}{\otimes} \mathcal{D}_k(M) \to \operatorname{Tor}_k^{\mathcal{A}}(\mathbf{F}_2, M), 1 \underset{\mathcal{A}}{\otimes} [Z] \mapsto [1 \underset{\mathcal{A}}{\otimes} Z]$.

The following definition uses this homomorphism for $M = \Sigma^{1-k}\mathbf{F}_2$ together with the suspension Σ and the desuspension Σ^{-1}.

Definition 2. [10]

$$\varphi_k^* := \Sigma^{-1} i_k (1 \underset{\mathcal{A}}{\otimes} \alpha_k^{-1}) \Sigma : \mathbf{F}_2 \underset{\mathcal{A}}{\otimes} D_k \to \operatorname{Tor}_k^{\mathcal{A}}(\mathbf{F}_2, \Sigma^{-k}\mathbf{F}_2)\,.$$

Remark. In Theorem 1, we also denote by φ_k^* the composite of the above φ_k^* with the suspension isomorphism $\Sigma^k : \operatorname{Tor}_{k,i}^{\mathcal{A}}(\mathbf{F}_2, \Sigma^{-k}\mathbf{F}_2) \xrightarrow{\cong} \operatorname{Tor}_{k,k+i}^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$.

We now prepare some materials for proving Theorem 1.

Let T_k be the Sylow 2-subgroup of GL_k consisting of all upper triangular $k \times k$-matrices with 1 on the main diagonal. The T_k-invariant ring, $M_k = P_k^{T_k}$, is called the Mùi algebra. In [12], Mùi shows that $P_k^{T_k} = \mathbf{F}_2[V_1, \ldots, V_k]$, where

$$V_i = \prod_{\lambda_j \in \mathbf{F}_2} (\lambda_1 x_1 + \cdots + \lambda_{i-1}x_{i-1} + x_i).$$

Then the Dickson invariant, $Q_{k,i}$, can inductively be defined by $Q_{k,i} = Q_{k-1,i-1}^2 + V_k \cdot Q_{k-1,i}$, where, by convention, $Q_{k,k} = 1$ and $Q_{k,i} = 0$ for $i < 0$. In [14], Singer sets $v_1 = V_1$, $v_k = V_k / V_1 \cdots V_{k-1}$ $(k \geq 2)$, so that

$$V_k = v_1^{2^{k-2}} v_2^{2^{k-3}} \cdots v_{k-1} v_k \quad (k \geq 2).$$

As $D_k \subset \mathbf{F}_2[v_1, \ldots, v_k]$, every element $q \in D_k$ has a unique expansion

$$q = \sum_{(j_1, \ldots, j_k)} v_1^{j_1} \cdots v_k^{j_k}\,,$$

where $j_1, \ldots, j_k$ are non-negative. We associate with $q \in D_k$ the following element:

Definition 3.

$$\tilde{q} = \sum_{(j_1, \ldots, j_k)} Sq^{j_1+1} \otimes \cdots \otimes Sq^{j_k+1} \otimes \Sigma^{1-k}1 \in B_{k-1}(\Sigma^{1-k}\mathbf{F}_2),$$

where $B_*(M)$ denotes the bar resolution of M over $\mathcal{A}$, for M a left $\mathcal{A}$-module.

Let $\partial_* : \mathcal{D}_k(\Sigma^{1-k}\mathbf{F}_2) := H_k(\mathcal{D}B_*(\Sigma^{1-k}\mathbf{F}_2)) \xrightarrow{\cong} H_{k-1}(EB_*(\Sigma^{1-k}\mathbf{F}_2))$ be the connecting isomorphism associated to the short exact sequence

$$0 \to EB_*(\Sigma^{1-k}\mathbf{F}_2) \to B_*(\Sigma^{1-k}\mathbf{F}_2) \to \mathcal{D}B_*(\Sigma^{1-k}\mathbf{F}_2) \to 0.$$

Proposition 1. *If $q \in D_k$, then $\tilde{q}$ is a cycle in $EB_{k-1}(\Sigma^{1-k}\mathbf{F}_2))$. Furthermore,*

$$\partial_*[1 \otimes \tilde{q}] = [\tilde{q}].$$

The next proposition deals with the isomorphism α_k treated in Theorem 2. It is actually an exposition of the Adem relations.

Proposition 2. *If $q \in D_k$, then*

$$\alpha_k[\tilde{q}] = \Sigma q.$$

Theorem 1 is proved by means of the above two propositions. To this end, using Singer's isomorphism $\Gamma_k^\wedge \cong \Lambda_k^*$, we need an explicit homotopy equivalence between the dual of the Lambda algebra and the bar resolution $B_*(\mathbf{F}_2)$. This is given by Priddy [13] as follows:

$$\begin{aligned} \Lambda_k^* &\to B_k(\mathbf{F}_2) \\ (\lambda_{j_1}\cdots\lambda_{j_k})^* &\mapsto 1 \otimes Sq^{j_1+1} \otimes \cdots \otimes Sq^{j_k+1} \otimes 1. \end{aligned}$$

The canonical $\mathcal{A}$-action on D_k is extended to an $\mathcal{A}$-action on $\Gamma_k^\wedge$. This action commutes with ∂_k (see [14]), so it determines an $\mathcal{A}$-action on $\operatorname{Ker}\partial_k$, the submodule of all cycles in $\Gamma_k^\wedge$. We also prove

Theorem 3. *φ_k^* factors through $\mathbf{F}_2 \underset{\mathcal{A}}{\otimes} \operatorname{Ker}\partial_k$ as shown in the commutative diagram:*

$$\begin{array}{ccc} \mathbf{F}_2 \underset{\mathcal{A}}{\otimes} D_k & \xrightarrow{\varphi_k^*} & \operatorname{Tor}_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2) \\ \bar{i} \searrow & & \nearrow \overline{pr} \\ & \mathbf{F}_2 \underset{\mathcal{A}}{\otimes} \operatorname{Ker}\partial_k\,, & \end{array}$$

where $\bar{i}$ is induced by the inclusion $D_k \subset \operatorname{Ker}\partial_k$ and $\overline{pr}$ is an epimorphism induced by the canonical projection $pr : \operatorname{Ker}\partial_k \to H_k(\Gamma^\wedge) \cong \operatorname{Tor}_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$.

In [7], we have stated the following conjecture: $D_k^+ \subset \mathcal{A}^+ \cdot \operatorname{Ker}\partial_k$ for $k > 2$. Obviously, this is stronger than Conjectures 2 and 3 and equivalent to the following:

Conjecture 4. *The homomorphism $\bar{i} : \mathbf{F}_2 \underset{\mathcal{A}}{\otimes} D_k \to \mathbf{F}_2 \underset{\mathcal{A}}{\otimes} \operatorname{Ker}\partial_k$, induced by the inclusion $i : D_k \to \operatorname{Ker}\partial_k$, is trivial for $k > 2$.*

Based on the above discussion, we believe that the problem of determining $\mathbf{F}_2 \underset{\mathcal{A}}{\otimes} \operatorname{Ker}\partial_k$ is something of interest.

The results of this note will be published in detail elsewhere.

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