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Short Communication

Complex Stability Radius of Linear Retarded Systems

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In this paper, we shall study the notion of complex stability radius of a stable linear time-delay system under structured perturbations. A lower bound and an upper bound are obtained which, in certain cases, yield a formula of the complex stability radius expressed in terms of the transfer function. Our results extend the results of Hinrichsen and Pritchard in [3] where linear systems with no delay $\dot{x} = Ax$ were considered.

Consider the linear retarded system described by a time-delay differential equation of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t - h_i), t \geq 0, \tag{1}$$

where $h_0 := 0 < h_1 < h_2 < \cdots < h_m$ are given positive numbers and $A_i \in \mathbf{C}^{n \times n}$, $i \in M := \{0, 1, \ldots, m\}$ are given complex matrices. Denote by $\sigma((A_i)_{i \in M})$ the set of all roots of the characteristic equation of the system (1):

$$\sigma((A_i)_{i \in M}) = \left\{ \lambda \in \mathbf{C} : \det \left(\lambda I - A_0 - \sum_{i=1}^{m} A_i e^{\lambda h_i} \right) = 0 \right\}.$$

The system (1) is said to be *asymptotically stable* if

$$\sigma((A_i)_{i \in M}) \subset \mathbf{C}^- := \{s \in \mathbf{C} : \operatorname{Re} s < 0\}.$$

Suppose the system (1) is asymptotically stable and the system matrices A_i are subjected to parameter perturbations of the form

$$A_i \to A_i + D_i \Delta_i E_i, \; i \in M. \tag{2}$$

Here, for each $i \in M$, l_i, q_i are given positive integer numbers, $D_i \in \mathbf{C}^{n\times l_i}$, $E_i \in \mathbf{C}^{q_i\times n}$ matrices defining the structure of the perturbations, and $\Delta_i \in \mathbf{C}^{l_i\times q_i}$ the unknown disturbance matrix whose size is measured by the norm $\|\Delta_i\|$. Throughout this paper, the norm of matrices is understood as the operator norm.

Generalizing the definition in [3], we define the *stability radius* of the asymptotically stable retarded system (1) with respect to structured perturbations (2) by

$$r_{\mathbf{C}} = \inf\left\{\sum_{i=0}^{m}\|\Delta_i\| \;:\; \Delta_i \in \mathbf{C}^{l_i\times q_i},\;\; i\in M,\;\; \sigma((A_i + D_i\Delta_i E_i)_{i\in M}) \not\subset \mathbf{C}^-\right\}. \tag{3}$$

We set, in the above definition, $\inf\emptyset = +\infty$.

For $i, j \in M$, we define the *associated transfer matrices* $G_{ij}(s) \in \mathbf{C}^{q_i\times l_j}$ by posing

$$G_{ij}(s) = E_i\left(sI - A_0 - \sum_{k=1}^{m} A_k e^{-sh_k}\right)^{-1} D_j \tag{4}$$

which are analytic matrix functions on every open subset of $\mathbf{C}\backslash\sigma((A_i)_{i\in M})$. We need the following:

Lemma 1. *Assume* (1) *is asymptotically stable. Then, for each* $i, j \in M$,

$$\max_{\mathrm{Re}\, s\geq 0}\|G_{ij}(s)\| = \max_{\mathrm{Re}\, s=0}\|G_{ij}(s)\|. \tag{5}$$

Proof. Since, by assumption, $\sigma((A_i)_{i\in M}) \subset \mathbf{C}^-$, it follows that, for each $y^* \in (\mathbf{C}^{q_i})^*$ and $u \in \mathbf{C}^{l_j}$, the function $s \mapsto y^*G_{ij}(s)u$ is analytic on $\mathbf{C}^+ := \{s \in \mathbf{C} : \mathrm{Re}\, s > 0\}$ and continuous on $\overline{\mathbf{C}^+} := \{s \in \mathbf{C} : \mathrm{Re}\, s \geq 0\}$. Since $\lim_{|s|\to\infty} |y^*G_{ij}(s)u| = 0$ (because $\lim \|(sI - A_0 - \sum_{k=1}^{m} A_k e^{-sh_k})^{-1}\| = 0$ (see [1])), by the maximum principle for the modulus of analytic functions, we obtain

$$\max_{\mathrm{Re}\, s\geq 0}|y^*G_{ij}(s)u| = \max_{\mathrm{Re}\, s=0}|y^*G_{ij}(s)u|.$$

Moreover, since

$$\|G_{ij}(s)\| = \sup_{\|u\|=1, \|y^*\|=1}|y^*G_{ij}(s)u|,$$

the last equality implies (5), completing the proof. ■

We now prove the main result which establishes the lower bound and the upper bound for the stability radius of system (1).

Theorem 2. *Assume the linear retarded system (1) is asymptotically stable and subjected to parameter perturbations of the form (2). If the complex stability radius of the system is defined by (3), then*

$$\frac{1}{\max_{i,j\in M,\omega\in R}\|G_{ij}(\imath\omega)\|} \leq r_{\mathbf{C}} \leq \frac{1}{\max_{i\in M,\omega\in R}\|G_{ii}(\imath\omega)\|} \tag{6}$$

Proof. First suppose $\Delta_i \in \mathbf{C}^{l_i \times q_i}$, $i \in M$ are destabilizing disturbances for (1). Then, by definition, there exist $x_0 \in \mathbf{C}^n$, $x_0 \neq 0$, $s_0 \in \mathbf{C}$ with $\operatorname{Re} s_0 \geq 0$ such that

$$\Big(A_0 + D_0 \Delta_0 E_0 + \sum_{j=1}^{m} (A_j + D_j \Delta_j E_j) e^{-s_0 h_j}\Big) x_0 = s_0 x_0. \tag{7}$$

Denote by $R(s)$ the characteristic matrix of the system (1):

$$R(s) = \left(sI - A_0 - \sum_{j=1}^{m} A_j e^{-s h_j} \right)^{-1}.$$

Then $R(s_0)$ is well defined since the system is stable and by (7),

$$R(s_0) D_0 \Delta_0 E_0 x_0 + \sum_{j=1}^{m} R(s_0) D_j \Delta_j E_j e^{-s_0 h_j} x_0 = x_0.$$

Let $q \in M$ be such an index such that $\|E_q x_0\| = \max_{j \in M} \|E_j x_0\|$. Then from the last equality, it follows that $E_q x_0 \neq 0$. Multiplying this equality from the left by E_q, we can deduce, by definition (4),

$$\|G_{q0}(s_0)\| \|\Delta_0\| \|E_0 x_0\| + \sum_{j=1}^{m} \|G_{qj}(s_0)\| \|\Delta_j\| \|E_j x_0\| \geq \|E_q x_0\|. \tag{8}$$

This implies

$$\left(\max_{i,j \in M} \|G_{ij}(s_0)\| \right) \left(\sum_{j=0}^{m} \|\Delta_j\| \right) \|E_q x_\Theta\| \geq \|E_q x_0\|.$$

Hence,

$$\sum_{j=0}^{m} \|\Delta_j\| \geq \frac{1}{\max_{i,j \in M} \|G_{ij}(s_0)\|} \geq \frac{1}{\max_{i,j \in M, \operatorname{Re} s \geq 0} \|G_{ij}(s)\|},$$

and thus, by Lemma 1,

$$r_{\mathbf{C}} \geq \frac{1}{\max_{i,j \in M, \omega \in \mathbb{R}} \|G_{ij}(\iota\omega)\|}.$$

To prove the second inequality in (6), we fix $i \in M$ and suppose the maximum of $\omega \mapsto \|G_{ii}(\iota\omega)\|$ occurs at $\omega_i \in \mathbf{R}$: $\max_{\omega \in \mathbb{R}} \|G_{ii}(\iota\omega)\| = \|G_{ii}(\iota\omega_i)\|$. Then there exists $u_i \in \mathbf{C}^{l_i}$, $\|u_i\| = 1$ such that $\|G_{ii}(\iota\omega_i)\| = \|G_{ii}(\iota\omega_i) u_i\|$. By the Hahn–Banach Theorem, there exists $f_i \in (C^{q_i})^*$, $\|f_i\| = 1$ such that

$$f_i(G_{ii}(\iota\omega_i) u_i) = \|G_{ii}(\iota\omega_i) u_i\| = \|G_{ii}(\iota\omega_i)\|.$$

Define the disturbance matrix $\Delta_i \in \mathbf{C}^{l_i \times q_i}$ by setting

$$\Delta_i = \|G_{ii}(\iota\omega_i)\|^{-1} u_i f_i.$$

It is clear that Δ_i is of rank one and

$$\|\Delta_i\| = \|G_{ii}(\iota\omega_i)\|^{-1}. \tag{9}$$

If we set

$$x = \left(\iota\omega_i I - A_0 - \sum_{j=1}^{m} A_j e^{-\iota\omega_i h_j}\right)^{-1} D_i u_i,$$

then $E_i x = G_{ii}(\iota\omega_i)u_i$, and hence, $\Delta_i E_i x = u_i$. It implies $x \neq 0$ and

$$x = \left(\iota\omega_i I - A_0 - \sum_{j=1}^{m} A_j e^{-\iota\omega_i h_j}\right)^{-1} D_i \Delta_i E_i x$$

or

$$\left(\iota\omega_i I - A_0 - D_i\Delta_i E_i - \sum_{j=1}^{m} A_j e^{-\iota\omega_i h_j}\right) x = 0.$$

Setting $\widetilde{\Delta}_j = 0$ $(\forall j \neq i)$ and $\tilde{\Delta}_i = \Delta_i e^{\iota\omega_i h_i}$, we obtain

$$\iota\omega_i \in \sigma((A_j + D_j\tilde{\Delta}_j E_j)_{j\in M}),$$

which means that $\widetilde{\Delta}_j$, $j = 0, 1, \ldots, m$ are destabilizing perturbations. By definition and (9), we have

$$r_{\mathbf{C}} \leq \sum_{j=0}^{m} \|\tilde{\Delta}_j\| = \|\Delta_i\| = \frac{1}{\|G_{ii}(\iota\omega_i)\|},$$

which implies the second inequality in (6) since the above inequality holds for every fixed $i \in M$. This concludes the proof. ∎

Corollary 3. *If $D_j = D$, $(\forall j \in M)$ or $E_j = E$ $(\forall j \in M)$, then we have the following formula for complex stability radius of the retarded system (1) under the structured perturbations of the form (2):*

$$r_{\mathbf{C}} = \frac{1}{\max_{i\in M,\omega\in\mathbb{R}} \|G_{ii}(\iota\omega)\|}. \tag{10}$$

We note that for the particular case where $D_j = D$ and $E_j = E$ for all $j \in M$, the above formula has been presented earlier in [4]. The proof given in [4], however, is incomplete.

Example. Consider the time-delay differential equation $\dot{x}(t) = px(t) + qx(t-1)$ with $0 \leq q \leq 1$ and $p < -1$. It is easily verified that this equation is asymptotically stable (see [1, Theorem 13.8]). Let the above equation be subjected to perturbations of the form

$$\dot{x}(t) = (p + \delta_1)x(t) + (q + \delta_2)x(t-1), \quad \delta_i \in \mathbf{C}. \tag{11}$$

Then, by (10), we have

$$r_{\mathbf{C}} = \frac{1}{\max_{\omega \in \mathbb{R}} |(\iota\omega - p - qe^{-\iota\omega})^{-1}|}$$
$$= \min_{\omega \in \mathbb{R}} \sqrt{(\omega + q\sin\omega)^2 + (p + q\cos\omega)^2} = -p - q.$$

It follows, in particular, that the perturbed equation (11) is asymptotically stable for all $\delta_1, \delta_2 \in \mathbf{C}$ such that $|\delta_1| + |\delta_2| < -p - q$. However, if $\delta_1 = -p - q$ and $\delta_2 = 0$, then the perturbed equation (11) is not asymptotically stable. ∎

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