Vietnam Journal of Mathematics **26**:4 (1998) 385–389

Vietnam Journal of MATHEMATICS

Short Communication

A Hopf-type Formula for $\dfrac{\partial u}{\partial t} + H(t,\, u,\, D_x u) = 0$*

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Received September 5, 1998

In a recent paper [4], Barron, Jensen, and Liu found the explicit viscosity solution of the problem

$$u_t + H(u, Du) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^n, \quad u(T, x) = g(x), \quad x \in \mathbb{R}^n$$

to be given by

$$u(t, x) = \inf \big\{\gamma \in \mathbb{R} : \sup_{p \in \mathbb{R}^n} (p.x - g^*(\gamma, p) + (T - t)H(\gamma, p)) \le 0\big\}, \tag{1}$$

where

$$g^*(\gamma, p) = \sup\{p.x : x \in \mathbb{R}^n, g(x) \le \gamma\}, \quad \gamma \in \mathbb{R}, \ p \in \mathbb{R}^n$$

is the *first quasiconvex conjugate* of the terminal function $g(x)$, $x \in \mathbb{R}^n$. This is an important result generalizing the Hopf formula while the Hamiltonian H depends on u and the terminal function may not be convex.

In this note, we consider the Cauchy problem for Hamilton–Jacobi equations, where the Hamiltonians depend on t, u and $D_x u$, namely,

$$\frac{\partial u}{\partial t} + H(t, u, D_x u) = 0, \ (t, x) \in \Omega := (0, T) \times \mathbb{R}^n, \tag{2}$$

$$u(0, x) = g(x), \ x \in \mathbb{R}^n. \tag{3}$$

*This work was supported in part by the National Basic Research Program in Natural Sciences, Vietnam.

Under the suitable assumptions, the viscosity solution is given by

$$u(t,x) = \inf\{\gamma \in \mathbb{R} : \sup_{p\in\mathbb{R}^n} (\langle p, x\rangle - g^*(\gamma, p) - \int_0^t H(\tau, \gamma, p)d\tau) \le 0\},$$
$$(t,x) \in \Omega. \tag{4}$$

Formula (1) in [4] was proved to be a viscosity solution under the assumption that the Hamiltonian $H(\gamma, p)$ is Lipschitz continuous in the variable p. Here, apart from the t dependence in the Hamiltonian $H(t, \gamma, p)$, we remove this assumption and require that the initial function g is quasiconvex and has *L-lsc property*. We will show later that the class of functions having L-lsc property contains the continuous functions f which are convex or strictly quasiconvex, and satisfy the following *growth condition*

$$f(x) \to +\infty \ \text{ as } |x| \to +\infty. \tag{5}$$

We first recall the definition of the quasiconvex dual according to the point of view of [2–4]. We also refer to [7] for multifunctions, [5, 6, 9] for viscosity solutions, and [1, 8, 10, 12, 13] for the Hopf formula. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. For any $\gamma \in \mathbb{R}$, denote

$$E_{f,\gamma} := \{x \in \mathbb{R}^n : f(x) \le \gamma\},$$
$$\operatorname{Arg\,min} f := \{x_0 \in \mathbb{R}^n : f(x_0) \le f(x), \quad \forall x \in \mathbb{R}^n\}.$$

A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *quasiconvex* if $E_{f,\gamma}$ is a convex set in $\mathbb{R}^n$ for any $\gamma \in \mathbb{R}$. Equivalently, f is quasiconvex if

$$f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\}, \quad 0 \le \lambda \le 1, \quad x, y \in \mathbb{R}^n.$$

The function f is said to be *strictly quasiconvex* if

$$f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}, \quad 0 < \lambda < 1, \quad \forall x \ne y.$$

Given a quasiconvex function f on $\mathbb{R}^n$, the *first quasiconvex conjugate of* f is defined by

$$f^*(\gamma, p) := \sup\{\langle p, x\rangle : x \in \mathbb{R}^n, f(x) \le \gamma\}, \quad \gamma \in \mathbb{R}, p \in \mathbb{R}^n.$$

If $\{x : f(x) \le \gamma\} = \emptyset$ for some γ, then $f^*(\gamma, p) = -\infty$. The *second conjugate of* f is defined by

$$f^{*\#}(x) := \inf\{\gamma \in \mathbb{R} : \sup_{p\in\mathbb{R}^n} (\langle p, x\rangle - f^*(\gamma, p)) \le 0\}.$$

Let f be a function defined on $\mathbb{R}^n$. Set $\gamma^* = \inf_{x\in\mathbb{R}^n} f(x)$. Consider the multifunction

$$L : (\gamma^*, +\infty) \to 2^{\mathbb{R}^n} \setminus \emptyset$$
$$\gamma \mapsto E_{f,\gamma},$$

which will be accordingly called the *generated multi* (by f).

Definition. *The function f is said to have L-lsc property if the generated multi L is $\varepsilon - \delta$-lsc.*

If a continuous function f satisfies the growth condition (5), then $E_{f,\gamma}$ is a compact set for each $\gamma \in \mathbb{R}$. This means that the multi L has compact values. Therefore, by virtue of Proposition 2.1 in [7], we obtain

Proposition 1. *If f is continuous and satisfies the growth condition (5), then the multi L is $\varepsilon - \delta$-lsc if and only if L is lsc.*

The class of functions having L-lsc property may be described by the next proposition. Its proof can be found in [14].

Proposition 2. *Let the continuous function f not attain its local minimum in any open subset of $\mathbb{R}^n \setminus \mathrm{Arg\,min}\, f$, and let f satisfy the growth condition (5). Then f has L-lsc property.*

It is known that a local minimum point of a strictly quasiconvex function must be a (unique) global minimum point (see [11, Chapter 9]). Of course, this statement also holds true for convex functions. Hence, from Proposition 2, we obtain the following:

Corollary 1. *Given a continuous function f satisfying (5), assume either f is strictly quasiconvex or f is convex. Then f has L-lsc property.*

Apart from that, an example of nonquasiconvex functions having L-lsc property can be regarded as

$$\xi(x) := \begin{cases} \cos x & \text{if } |x| < 3\pi/2, \\ |x| - 3\pi/2 & \text{if } |x| \geq 3\pi/2. \end{cases}$$

We are considering the Cauchy problem (2)–(3). The following conditions will be imposed upon the Hamiltonian H and the initial data.

(A) The initial function $g \in C(\mathbb{R}^n)$ is quasiconvex, has L-lsc property, and satisfies the growth condition

$$g(x) \to +\infty \text{ as } |x| \to +\infty. \tag{6}$$

(B) The Hamiltonian $H : [0, T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous and
- (i) $H(t, \gamma, \lambda p) = \lambda H(t, \gamma, p)$ for all $(t, \gamma, p) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n$, $\lambda \geq 0$;
- (ii) $H(t, \gamma, p)$ is nondecreasing in $\gamma \in \mathbb{R}$ for each $(t, p) \in [0, T] \times \mathbb{R}^n$.

(C) The Hamiltonian H satisfies one of the following two:
- (i) To every fixed $t_0 \in (0, T)$, there exists a function $h : [0, T] \times \mathbb{R} \to \mathbb{R}$, $h(t, \gamma)$ is positive for almost every $t \in (0, T)$, and $h(., \gamma)$ is integrable for any γ, such that
$$H(t, \gamma, p) = h(t, \gamma) H(t_0, \gamma, p), \ \forall (t, \gamma, p) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n.$$
- (ii) If $0 \leq \alpha_i \leq 1$, $|p_i| = 1$, $i = 1, \ldots, m$, and $\sum_{i=1}^{m} \alpha_i = 1$, then
$$H(t, \gamma, \sum_{i=1}^{m} \alpha_i p_i) \geq \sum_{i=1}^{m} \alpha_i H(t, \gamma, p_i),$$
for all $(t, \gamma) \in [0, T] \times \mathbb{R}$.

Our main result is given by

Theorem 1. *Under the hypotheses (A)–(C), the formula (4) determines a viscosity solution of problem (2)–(3):*

$$u(t,x) = \inf\Big\{\gamma \in I\!R : \sup_{p\in\mathbb{R}^n}\big(\langle p,x\rangle - g^*(\gamma,p) - \int_0^t H(\tau,\gamma,p)d\tau\big) \le 0\Big\},$$

$$(t,x)\in\Omega.$$

As an immediate consequence of Corollary 1 and Theorem 1, we have

Corollary 2. *Let (B)–(C) hold, and let g be continuous and satisfy the growth condition (6). In addition, suppose that g is either strictly quasiconvex or convex. Then formula (4) determines a viscosity solution of (2)–(3).*

Example. Consider the following Cauchy problem:

$$\frac{\partial u}{\partial t} - (1+t)^{-u}|D_x u| = 0, \ \ (t,x)\in\Omega, \tag{7}$$

$$u(0,x) = |x|, \ \ x\in\mathbb{R}^n. \tag{8}$$

Evidently, all the assumptions of Corollary 2 are fulfilled. Hence, problem (7)–(8) has the viscosity solution given by the Hopf-type formula (4) as follows, for $(t,x)\in\Omega$,

$$u(t,x) = \gamma_0,$$

where $\gamma_0 \ge 0$ is the unique solution of the equation $\gamma - \int_0^t (1+\tau)^{-\gamma}d\tau - |x| = 0.$

The proof of Theorem 1 can be found in [14].

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