Vietnam Journal of Mathematics **27**:1 (1999) 69–84

Vietnam Journal
of
MATHEMATICS

Majorized Powers of an Operator, Discrete Orbits and Hyperinvariant Subspaces

Ralph deLaubenfels[1] and Vu Quoc Phong[2]

[1] *Scientia Research Institute, P.O. Box 988, Athens, Ohio 45701, USA*

[2] *Department of Mathematics, Ohio University, Athens, Ohio 45701, USA*

Received April 21, 1998

Abstract. For any weight sequence $\vec{\alpha} = \{\alpha(k)\}_{k \in Z}$ and a closed linear operator G on a Banach space X, we construct a maximal continuously embedded Banach subspace $Z_{\vec{\alpha}}(G)$ such that $\|(G|_{Z_{\vec{\alpha}}(G)})^k\| \leq \alpha(k)$, for all $k \in Z$. We use this to produce many hyperinvariant subspaces for operators with an appropriate orbit $\{G^k x\}_{k \in \mathbf{Z}}$ or one-sided orbit $\{G^n x\}_{n \in \mathbf{N}}$.

1. Introduction and Terminology

We cannot begin to summarize the literature on invariant subspaces for linear operators on a Banach space, so we will refer the reader to [3] and the references therein.

It is convenient to specify some terminology before continuing.

1.1. Terminology

All operators are linear on a Banach space. The letters X, W and Z, among others, will always represent Banach spaces. We will denote by $\mathcal{D}(A)$ the domain of the operator A, by $\text{Im}(A)$ its image, by $\mathcal{N}(A)$ its null space, by $\sigma(A)$ its spectrum, and by $\rho(A)$ its resolvent set. Denote by $B(X)$ the space of bounded operators from X to itself.

We will say that X is continuously embedded in W, $X \hookrightarrow W$, if $X \subseteq W$ and the identity map from X into W are continuous. If $B \in B(W)$, then $B|_X$ is the part of B in X, that is, $\mathcal{D}(B|_X) \equiv \{x \in X \cap \mathcal{D}(B) \mid Bx \in X\}$, with $(B|_X)x \equiv Bx$, for all $x \in \mathcal{D}(B|_X)$. If G is an operator on X and B is an operator on W, with $G = B|_X$, we will then say that G is continuously embedded in B.

Assume throughout this paper that G is a closed operator on X.

The subspace $W \subseteq X$ is *invariant* for G if G maps $W \cap \mathcal{D}(G)$ into W. The space W is *hyperinvariant* for G if it is invariant for R, whenever $R \in B(X)$ commutes with G, that is, $RG \subseteq GR$. We will say the (hyper-)invariant subspace W for G is *nontrivial* if $\mathcal{D}(G) \cap W$ is neither $\mathcal{D}(G)$ nor $\{\vec{0}\}$.

Decomposable operators come with many closed hyperinvariant subspaces, namely, their local spectral subspaces. Operators whose powers (both positive and negative) satisfy appropriate growth conditions are automatically decomposable.

If $\vec{\alpha}$ is a Beurling sequence (see Definition 4.1) and

$$\|G^k\| = O(\alpha(k)) \ \forall k \in \mathbf{Z}, \tag{1}$$

then G has a functional calculus $f \mapsto f(G)$ defined by

$$f(G) \equiv \sum_{k \in \mathbf{Z}} \hat{f}(k) G^k,$$

for f in $C^\infty(\Upsilon)$, where Υ is the unit circle, whose Fourier series $\{\hat{f}(k)\}_{k\in\mathbf{Z}}$ decays sufficiently rapidly, and the local spectral subspaces have an explicit form

$$E(\Omega) \equiv \cap\{\mathcal{N}(f(G)) \mid f \in C^\infty(\Upsilon) \ \text{ has support disjoint from } \Omega\} \tag{2}$$

(see [4]).

We show that, for G closed (not necessarily bounded), the set of all x, for which the discrete orbit $\{G^k x\}_{k\in\mathbf{Z}}$ satisfies the growth condition

$$\|G^k x\| = O(\alpha(k)) \ \ (k \in \mathbf{Z}),$$

can be normed in such a way as to form a Banach space, $Z_{\vec{\alpha}}(G)$, continuously embedded in X, on which G satisfies (1).

Thus, when G has a nontrivial discrete orbit $\{G^k x\}_{k\in\mathbf{Z}}$ that grows like a Beurling sequence, we may continuously embed a decomposable operator in G. When G^* has such an orbit, we may, after taking adjoints, continuously embed G in a decomposable operator. On both sides of G, there are numerous hyperinvariant subspaces with the explicit form (2). This will produce closed nontrivial hyperinvariant subspaces for G.

It is sometimes sufficient to consider only one-sided orbits $\{G^n x\}_{n\in\mathbf{N}}$ by using another construction of a decomposable operator in which G is continuously embedded (see Lemma 4.4).

In Sec. 2, we show that, if G is continuously embedded in a decomposable operator (on a larger space with a weaker norm) B, then G inherits a family of closed invariant subspaces formed by taking the intersection of X with the local spectral subspaces of B. In order that one of these be nontrivial, it is sufficient that there exist disjoint closed subsets of the complex plane, Ω_1 and Ω_2, such that the local spectral subspaces for G corresponding to Ω_1 and Ω_2 are both more than the zero vector.

In Sec. 3, we construct, for any weight sequence $\vec{\alpha}$, the maximal continuously embedded subspace of X, $Z_{\vec{\alpha}}(G)$, on which the powers of G are dominated by $\vec{\alpha}$, as in (1).

Section 4 applies Secs. 2 and 3 to produce simple sufficient conditions on the orbits of G and G^* in order for G to have a nontrivial closed hyperinvariant subspace. For the dual G^* of a $C_{1,1}$ contraction (see Definition 4.12), we construct a family of closed hyperinvariant subspaces with a "pointwise" version of (2); for Ω, a closed subset of the unit circle Υ, define $E(\Omega)$ to be the set of all $x \in X$ such that

$$\lim_{N\to\infty} \overline{\lim}_{n\to\infty} \| \sum_{k=-N}^{N} \hat{f}(k)(G^*)^{k+n} x \| = 0,$$

whenever $f \in C^\infty(\Upsilon)$ has support disjoint from Ω (Corollary 4.13). A similar construction works for a much larger class of operators (see Theorem 4.11).

2. (Hyper-)Invariant Subspaces for Operators Continuously Embedded in a Decomposable Operator

Proposition 2.6 gives an idea of how large a family of closed invariant subspaces we can obtain for G, when it is continuously embedded between two decomposable operators. Theorem 2.5 is a generalization that only has G continuously embedded in a decomposable operator. Except for the hyperinvariance, Corollary 2.7 and Proposition 2.8 appeared in [7].

In all these results, "invariant" may be replaced by "hyperinvariant" when we have the following additional condition.

Definition 2.1. *Suppose W is a Banach space, $X \hookrightarrow W$, $B \in B(W)$, and $G = B|_X$. We will say that B satisfies the commuting condition for G if, whenever $R \in B(X)$, with $RG \subseteq GR$, then there exists $S \in B(W)$ such that $SB = BS$ and $R = S|_X$.*

The following definitions are from [9] (see also [4, 14–16].

Definition 2.2. *If $x \in X$, then a complex number λ_0 is in the local resolvent set, $\rho(G, x)$, of G, if there exists a neighborhood Ω, of λ_0, and a map $\lambda \mapsto R(\lambda, G, x)$, from Ω into $\mathcal{D}(G)$, such that*

$$(\lambda - G)R(\lambda, G, x) = x, \quad \forall \lambda \in \Omega.$$

The local spectrum $\sigma(G, x)$ is the complement, in $\mathbf{C}$, of $\rho(G, x)$.

If Ω is a closed subset of the complex plane, then the local spectral subspace corresponding to Ω [9] is

$$X_G(\Omega) \equiv \{x \in X \mid \sigma(G, x) \subseteq \Omega\}.$$

The operator $A \in B(X)$ is *decomposable* [9, Definition 5.1 and Corollary 6.5] if, whenever $\{\Omega_i\}_{i=1}^n$ is an open cover of $\sigma(A)$, then there exist subspaces $\{X_i\}_{i=1}^n$, invariant for A, such that

(1) $\sigma(A|_{X_i}) \subseteq \Omega_i \;\; 0 \leq i \leq n)$; and
(2) $X = \sum_{i=0}^n X_i$.

The following is from [7].

Lemma 2.3. *Suppose $Z \hookrightarrow W$ and B is a closed operator on W. Then, for any closed $\Omega \subseteq \mathbf{C}$, $x \in Z$,*

$$\sigma(B, x) \subseteq \sigma(B|_Z, x) \;\; and \;\; Z_{B|_Z}(\Omega) \subseteq W_B(\Omega).$$

Proposition 2.4. *Suppose $X \hookrightarrow W$, $G = B|_X$, and $B \in B(W)$ is decomposable. Define, for any closed $\Omega \subseteq \mathbf{C}$,*

$$E(\Omega) \equiv X \cap W_B(\Omega).$$

Then $\Omega \mapsto E(\Omega)$ has the following properties:
(1) *$E(\Omega)$ is closed and invariant under G.*
(2) *$E(\phi) = \{\vec{0}\}$, $E(\mathbf{C}) = X$.*
(3) *$X = E(\overline{O_1}) + \cdots + E(\overline{O_m})$, whenever $\{O_1, ..., O_m\}$ is an open cover of $\mathbf{C}$.*
(4) *$E(\bigcap_{k=1}^{\infty} \Omega_k) = \bigcap_{k=1}^{\infty} E(\Omega_k)$.*
(5) *$X_G(\Omega) \subseteq E(\Omega)$.*

If B satisfies the commuting condition for G, then $E(\Omega)$ is hyperinvariant under G.

Proof. Assertions (1)–(4) follow from the properties of $\Omega \mapsto W_B(\Omega)$ since B is decomposable. The hyperinvariance, when B satisfies the commuting condition for G, follows from the fact that $W_B(\Omega)$ is a hyperinvariant subspace for B. Assertion (5) is a consequence of Lemma 2.3. ■

Remark. We shall see that spaces W, as in Proposition 2.4, arise very easily. For example, if G is a C_1 contraction (this means that G is a contraction) and for any nontrivial x,

$$\lim_{n\to\infty} G^n x \neq 0,$$

and $\mathrm{Im}(G)$ is dense, then there exists a Banach space W, $B \in B(W)$ such that $X \hookrightarrow W, G = B|_X$, and B is an invertible isometry (see [18, Lemma 3.5] or [3, Chpt. XII.1]).

What is missing from (1)–(4) is any guarantee that $E(\Omega)$ is nontrivial. Condition (5) is all we have to create nontrivial $E(\Omega)$ as follows.

Theorem 2.5. *Suppose $X \hookrightarrow W, G = B|_X$, $B \in B(W)$ is decomposable and there exist disjoint closed Ω_1, Ω_2 such that $X_G(\Omega_j) \neq \{\vec{0}\}$, for $j = 1, 2$. Then $E(\Omega_j)$ is nontrivial for $j = 1, 2$ and $E(\Omega_1) \cap E(\Omega_2) = \{\vec{0}\}$.*

In particular, G then has a pair of nontrivial closed invariant subspaces with trivial intersection. If B satisfies the commuting condition for G, then these subspaces are hyperinvariant.

Remark and Examples. Theorem 2.5 is somewhat a local version of the usual sufficient condition for producing nontrivial closed hyperinvariant subspaces, having the spectrum of G separated. But the hypotheses of Theorem 2.5 are much more likely.

Consider, for example, $(Gf)(z) \equiv zf(z)$ on X defined to be one of the usual spaces of complex-valued functions on Ω, a subset of the complex plane, such as $BC(\Omega)$, $L^p(\Omega)(1 \leq p < \infty)$, a Sobolev space, etc. The hypotheses of Theorem 2.5 are equivalent to there being nontrivial functions $f_1, f_2 \in X$ with disjoint support Ω_1, Ω_2. The spectrum of G is separated if and only if Ω is not connected.

To see the limitations of Theorem 2.5, consider $(Gf)(z) \equiv zf(z)$ on $X \equiv H^\infty(D) \cap C(\overline{D})$, where D is the open unit disc. G is continuously embedded in $(Bf)(z) \equiv zf(z)$ on $W \equiv C(\partial D)$, which is clearly decomposable. However, there do not exist disjoints Ω_1, Ω_2 as in Theorem 2.5. In fact, for any closed $\Omega \subseteq \overline{D}$, $X_G(\Omega)$ is either $\{0\}$ or X; if $\Omega \neq \overline{D}$, then $X_G(\Omega) = \{0\}$.

When G is embedded between two decomposable operators, that is,

$$Z \hookrightarrow X \hookrightarrow W,$$

$G = B|_X$, and both B and $G|_Z$ are decomposable, then Lemma 2.3 implies that it is sufficient to apply Theorem 2.5 to have $Z_{G|_Z}(\Omega_j) \neq \{0\}$ for $j = 1, 2$. We would prefer to replace conditions on $G|_Z$ with conditions on G, for example, conditions on the local spectrum of G.

Proposition 2.6. *Suppose there exist nontrivial Banach spaces Z, W, $B \in B(W)$, such that*

$$Z \hookrightarrow X \hookrightarrow W,$$

$G = B|_X$, and both B and $G|_Z$ are decomposable, $x \in Z$, and there exist open O_1, O_2, V_1, V_2 such that

(1) $\overline{O_1}$ and $\overline{O_2}$ are disjoint;

(2) $\sigma(G, x) \subseteq O_j \cup V_j$, for $j = 1, 2$;

(3) $\sigma(G, x) - \overline{O_j}$ and $\sigma(G, x) - \overline{V_j}$ are nonempty, for $j = 1, 2$.

Then for $j = 1, 2$, $E(\overline{O_j})$, defined by Proposition 2.4, is nontrivial, and $E(\overline{O_1}) \cap E(\overline{O_2}) = \{\vec{0}\}$.

Proof. Since, by Lemma 2.3, $Z(\Omega) \equiv Z_{G|_Z}(\Omega) \subseteq X_G(\Omega)$, for any closed Ω, it is sufficient, by Theorem 2.5, to show that $Z(\overline{O_j}) \neq \{\vec{0}\}$ for $j = 1, 2$. By Lemma 2.3,

$$\sigma(G, x) \subseteq \sigma(G|_Z, x) \subseteq \sigma(G|_Z),$$

thus,

$$\sigma(G|_Z) \subseteq O_1 \cup U_1,$$

where $U_1 \equiv [V_1 \cup (\mathbf{C} - \sigma(G, x))]$. Since $G|_Z$ is decomposable, we have

$$Z = Z(\overline{O_1}) + Z(\overline{U_1}), \tag{$*$}$$

and

$$\sigma(G|_{Z(\overline{U_1})}) \subseteq \overline{U_1},$$

since

$$\sigma(G|_Z) - \overline{U_1} = \sigma(G, x) - \overline{V_1},$$

which is nonempty, it follows that $Z(\overline{U_1})$ is not all of Z, so that, by $(*)$, $Z(\overline{O_1}) \neq \{\vec{0}\}$. Identically, $Z(\overline{O_2}) \neq \{\vec{0}\}$. ■

Proposition 2.6 is capable of producing large families of nontrivial closed invariant subspaces, but it sounds technical. The following special case is a much simpler way to verify the existence of at least a pair of nontrivial closed invariant subspaces with trivial intersection.

Corollary 2.7. *Suppose there exist nontrivial Banach spaces Z, W, $B \in B(W)$, such that*

$$Z \hookrightarrow X \hookrightarrow W,$$

$G = B|_X$, and both B and $G|_Z$ are decomposable, $x \in Z$, and $\sigma(G, x)$ contains at least two points. Then G has a pair of nontrivial closed invariant subspaces whose intersection is trivial.

If B satisfies the commuting condition for G, then these subspaces are hyperinvariant.

Proof. Suppose $\lambda_j \in \sigma(G, x)$, $j = 1, 2$. Let $\epsilon \equiv \frac{1}{2}|\lambda_1 - \lambda_2|$, and define, for $j = 1, 2$,

$$O_j \equiv \left\{z \in \mathbf{C} \mid |z - \lambda_j| < \epsilon\right\}, \quad V_j \equiv \{z \in \mathbf{C} \mid |z - \lambda_j| > \frac{\epsilon}{2}\right\}.$$

Now, apply Proposition 2.6. ■

When $\sigma(G, x)$ is empty or a single point, for all $x \in Z$, we need additional information about $G|_Z$.

Proposition 2.8. *Suppose there exist nontrivial Banach spaces Z, W, decomposable $B \in B(W)$, such that*

$$Z \hookrightarrow X \hookrightarrow W,$$

$G = B|_X$, and $G|_Z$ is bounded and has a bounded inverse, such that, for some $m > 0$,

$$\|(G|_Z)^k\| = O(k^m) \text{ and } \|(G|_Z)^{-k}\| = e^{o(\sqrt{k})}, \text{ as } k \to +\infty.$$

Then G has a nontrivial closed invariant subspace. If B satisfies the commuting condition for G, then either G is a multiple of the identity or G has a nontrivial closed hyperinvariant subspace.

Proof. If $\sigma(G|_Z)$ is a set containing a single point $\{\lambda_0\}$, then by Corollary 3.5 in [10], λ_0 is an eigenvalue of G. Thus, either $G = \lambda_0 I$ or the eigenspace for G is a nontrivial closed hyperinvariant subspace for G.

If $\sigma(G|_Z)$ contains two or more points, note that, since $\{\|(G|_Z)^k\|\}_{k\in\mathbf{Z}}$ is dominated by a Beurling sequence, $G|_Z$ is decomposable (see Definition 4.1 and Lemma 4.2). Thus, this follows from Corollary 2.7. ■

3. Maximal Continuously Embedded Banach Subspaces on Which an Operator Has Majorized Powers

In this section, given a weight sequence $\vec{\alpha}$, we construct a maximal continuously embedded Banach subspace Z, of X, on which G is bounded, and has powers whose norms $\|(G|_Z)^k\|$ are $O(\alpha(k))$.

These spaces are in the spirit of Kantorovitz's semi-simplicity manifold [11, 12] and Hille–Yosida space [13]; the latter was introduced independently in [13]. We take a "pointwise" approach analogous to Chapter V in [5] and [6]. The special case of $\alpha(k) \equiv 1$ is in [8], where it was called a discrete Hille–Yosida space (see [7] for a semigroup analog of this section).

Definition 3.1. $C^\infty(G) \equiv \cap_{k=0}^\infty \mathcal{D}(G^k)$.

Definition 3.2. *A sequence $\vec{\alpha} \equiv \{\alpha(k)\}_{k\in\mathbf{Z}}$ (one-sided sequence $\vec{\alpha} \equiv \{\alpha(k)\}_{k=0}^\infty$) is a weight sequence (one-sided weight sequence) if $\alpha(0) \geq 1$ and*

$$\alpha(n + m) \leq \alpha(n)\alpha(m), \; \forall n, m \in \mathbf{Z} \; (n, m \in \mathbf{N} \cup \{0\}).$$

Definition 3.3. *If $\vec{\alpha}$ is a one-sided weight sequence, define $Z_{\vec{\alpha}}(G)$ to be the set of all $x \in C^{\infty}(G)$ such that*

$$\|x\|_{Z_{\vec{\alpha}}(G)} \equiv \sup_{k \in \mathbf{N} \cup \{0\}} \frac{1}{\alpha(k)} \|G^k x\| < \infty.$$

Theorem 3.4. *If $\vec{\alpha}$ is a one-sided weight sequence, then*

(1) *$Z_{\vec{\alpha}}(G)$ is a Banach space continuously embedded in X;*

(2) *$Z_{\vec{\alpha}}(G)$ is left invariant by G and*

$$\|(G|_{Z_{\vec{\alpha}}(G)})^m\|_{B(Z_{\vec{\alpha}}(G))} \leq \alpha(m), \quad \forall m \in \mathbf{N};$$

(3) *$Z_{\vec{\alpha}}(G)$ is maximal-unique, that is, if $W \hookrightarrow X$ is a Banach space such that $G|_W \in B(W)$, with $\|(G|_W)^m\| = O(\alpha(m))$ for $m \in \mathbf{N}$, then $W \hookrightarrow Z_{\vec{\alpha}}(G)$; and*

(4) *if $B \in B(X)$ and $BG \subseteq GB$, then B maps $Z_{\vec{\alpha}}(G)$ to itself, and*

$$\|B|_{Z_{\vec{\alpha}}(G)}\| \leq \|B\|.$$

Proof. (1) It is clear that $Z_{\vec{\alpha}}(G)$ is a normed vector space continuously embedded in X. To show completeness, suppose $\{x_n\}_n$ is Cauchy in $Z_{\vec{\alpha}}(G)$.

Define, for $n \in \mathbf{N}$, a vector $\vec{x}_n \in Y \equiv \ell^{\infty}(\mathbf{N} \cup \{0\}, X)$ by

$$(\vec{x}_n)_k \equiv \frac{1}{\alpha(k)} G^k x_n \ (k - 1 \in \mathbf{N}).$$

Then $\vec{x}_n$ is Cauchy, and hence, converges to $\vec{y} \in Y$.

For any nonnegative integer k, $G^k x_n \to \alpha(k) y_k$ and $G(G^k x_n) \to \alpha(k+1) y_{k+1}$, as $n \to \infty$, thus, since G is closed, it follows that $y_k \in \mathcal{D}(G)$ and

$$G y_k = \frac{\alpha(k+1)}{\alpha(k)} y_{k+1}. \tag{$*$}'$$

Let $x \equiv \alpha(0) y_0$. By $(*)'$ and induction, it follows that $x \in C^{\infty}(G)$ and

$$G^k x = \alpha(k) y_k \ (k - 1 \in \mathbf{N});$$

in other words,

$$\frac{1}{\alpha(k)} G^k (x_n - x) = [\vec{x}_n - \vec{y}]_k .$$

Thus, since $\vec{y} \in \ell^{\infty}$ and $\vec{x}_n \to \vec{y}$ in ℓ^{∞}, it follows that $x \in Z_{\vec{\alpha}}(G)$ and $x_n \to x$ in $Z_{\vec{\alpha}}(G)$. Thus, $Z_{\vec{\alpha}}(G)$ is complete.

(2) For any $m, k \in \mathbf{N} \cup \{0\}$, $x \in Z_{\vec{\alpha}}(G)$,

$$\frac{1}{\alpha(k)} \|G^k (G^m x)\| \leq \frac{\alpha(k+m)}{\alpha(k)} \|x\|_{Z_{\vec{\alpha}}(G)} \leq \alpha(m) \|x\|_{Z_{\vec{\alpha}}(G)},$$

thus, G maps $Z_{\vec{\alpha}}(G)$ to itself and

$$\|G^m x\|_{Z_{\vec{\alpha}}(G)} \le \alpha(m)\|x\|_{Z_{\vec{\alpha}}(G)},$$

for all $x \in Z_{\vec{\alpha}}(G)$, $m \in \mathbf{N}$.

(3) Suppose W is as indicated. There exists a constant M such that

$$\|G^m x\|_W \le M\alpha(m)\|x\|_W \quad \forall x \in W, m \in \mathbf{N}\cup\{0\}.$$

Since $W \hookrightarrow X$, there exists a constant $\delta > 0$ such that

$$\|x\|_W \ge \delta\|x\| \quad \forall x \in W.$$

Suppose $x \in W$. Since $G|_W \in B(W)$, it follows that $x \in C^\infty(G|_W) \subseteq C^\infty(G)$. For any $k \in \mathbf{N}\cup\{0\}$,

$$\frac{1}{\alpha(k)}\|G^k x\| \le \frac{1}{\delta\alpha(k)}\|G^k x\|_W \le \frac{M}{\delta}\|x\|_W,$$

thus,

$$\|x\|_{Z_{\vec{\alpha}}} \le \frac{M}{\delta}\|x\|_W;$$

this is to say $W \hookrightarrow Z_{\vec{\alpha}}(G)$.

(4) Suppose $x \in Z_{\vec{\alpha}}$. Then for any $k \in \mathbf{N}\cup\{0\}$,

$$\frac{1}{\alpha(k)}\|G^k Bx\| = \frac{1}{\alpha(k)}\|BG^k x\| \le \|B\|\left[\frac{1}{\alpha(k)}\|G^k x\|\right] \le \|B\|\|x\|_{Z_{\vec{\alpha}}},$$

thus,

$$\|Bx\|_{Z_{\vec{\alpha}}} \le \|B\|\|x\|_{Z_{\vec{\alpha}}}, \quad \forall x \in Z_{\vec{\alpha}},$$

as desired. ■

Remark. The space $C^\infty(G)$, with the seminorms

$$\|x\|_k \equiv \|G^k x\|$$

for k a nonnegative integer, is similarly the maximal Frechet space continuously embedded in X, on which G is bounded.

Definition 3.5. *Now, assume G is closed and injective, and $\vec{\alpha} \equiv \{\alpha(k)\}_{k\in\mathbf{Z}}$ is a weight sequence.*

Define $Z_{\vec{\alpha}}(G)$ to be the set of all $x \in C^\infty(G)\cap C^\infty(G^{-1})$ such that

$$\|x\|_{Z_{\vec{\alpha}}(G)} \equiv \sup_{k\in\mathbf{Z}} \frac{1}{\alpha(k)}\|G^k x\| < \infty.$$

The same proof as the proof of Theorem 3.4 gives us the following:

Theorem 3.6. *If $\vec{\alpha}$ is a weight sequence, then*

(1) *$Z_{\vec{\alpha}}(G)$ is a Banach space continuously embedded in X;*
(2) *$Z_{\vec{\alpha}}(G)$ is left invariant by G and G^{-1} and*

$$\|G^m\|_{B(Z_{\vec{\alpha}}(G))} \leq \alpha(m), \ \forall m \in \mathbf{Z};$$

(3) *$Z_{\vec{\alpha}}(G)$ is maximal-unique, that is, if $W \hookrightarrow X$ is a Banach space such that $G|_W$ and $G^{-1}|_W$ are in $B(W)$, with $\|(G|_W)^m\| = O(\alpha(m))$, for $m \in \mathbf{Z}$, then $W \hookrightarrow Z_{\vec{\alpha}}(G)$; and*
(4) *if $B \in B(X)$ and $BG \subseteq GB$, then B maps $Z_{\vec{\alpha}}(G)$ to itself, and*

$$\|B|_{Z_{\vec{\alpha}}(G)}\| \leq \|B\|.$$

4. Orbits and Hyperinvariant Subspaces

Without loss of generality, we may assume, throughout this section, that G is injective and $\mathrm{Im}(G)$ is dense; otherwise, $\overline{\mathrm{Im}(G)}$ or $\mathcal{N}(G)$ would provide a closed nontrivial hyperinvariant subspace for G.

Also, assume throughout this section that G is not a multiple of the identity operator.

We give sufficient conditions, in terms of orbits of G and G^*, for G to have a nontrivial closed hyperinvariant subspace. Theorem 4.3 requires that an orbit of G and an orbit of G^* be a Beurling sequence. In Theorem 4.3, G need not be bounded. Theorem 4.5 requires that there exist a one-sided Beurling sequence $\vec{\alpha}$ such that

$$\|G^n\| = O(\alpha(n)) \ (n \in \mathbf{N}) \ \text{ and } \ \overline{\lim}_{n\to\infty} \frac{1}{\alpha(n)} \|G^n x\| > 0, \tag{3}$$

and disjoint closed sets Ω_j, $j = 1, 2$, whose local spectral subspaces $X_G(\Omega_j)$ are nontrivial. Theorem 4.6 replaces the spectral subspace condition in Theorem 4.5 with the requirement that an orbit of G be a Beurling sequence. Theorem 4.8 requires (3) for both G and G^*. A special case ($\alpha(n) \equiv 1$) is G being a $C_{1,1}$ contraction. Theorem 4.11 constructs a large family of closed hyperinvariant subspaces for G as in Theorem 4.8, with a pointwise version of the construction of local spectral subspaces for generalized scalar operators.

Continuous analogs of Theorems 4.3, 4.6 and 4.8 (although not producing (hyper-)invariant subspaces) appear in [7].

Definition 4.1. *A Beurling sequence (one-sided Beurling sequence) is a weight sequence (one-sided weight sequence) $\vec{\alpha}$ such that*

$$\sum_{k\in Z} \frac{\ln \alpha(k)}{1+k^2} < \infty \ \left(\sum_{n=0}^{\infty} \frac{\ln \alpha(n)}{1+n^2} < \infty \right).$$

Lemma 4.2. *If $\vec{\alpha}$ is a Beurling sequence, then $G|_{Z_{\vec{\alpha}}(G)}$ is decomposable.*

Proof. This is an immediate consequence of Theorem 3.6(2) and [4, Chpt. 5.2]. ∎

Theorem 4.3. *Suppose $\mathcal{D}(G)$ is dense, and there exist nontrivial $x \in C^\infty(G) \cap C^\infty(G^{-1})$, $x^* \in C^\infty(G^*) \cap C^\infty((G^{-1})^*)$ such that*

$$\|G^k x\| = O(\alpha_1(k)), \quad \|(G^*)^k x^*\| = O(\alpha_2(k))$$

for some Beurling sequences $\vec{\alpha}_j$, $j = 1, 2$, and either

(1) *for some $m > 0$,*

$$\|G^n x\| = O(n^m) \text{ and } \|G^{-n} x\| = e^{o(\sqrt{n})} \text{ as } n \to +\infty,$$

or

(2) *$\sigma(G, x)$ contains at least two points.*

Then G has a nontrivial closed hyperinvariant subspace.

Proof. First, assume we are under hypothesis (1), and $\sigma(G, x)$ is empty or consists of a single point. Then by Corollary 3.5 in [10], G has an eigenvector, and we are done.

Now, suppose we are under hypothesis (2). In Corollary 2.7, let $Z \equiv Z_{\vec{\alpha}_1}(G)$. By Lemma 4.2, $G|_Z$ is decomposable. Let $Y \equiv Z_{\vec{\alpha}_2}(G^*)$. By Lemma 4.2 again, $(G^*)|_Y$ is decomposable. Let $W \equiv Y^*$, $B \equiv (G^*|_Y)^*$. Then $B \in B(W)$ is decomposable [9, Theorem 8.1].

If the closure of Y in X^* is not all of X^*, then by Theorem 3.6(4), this closure is a nontrivial closed hyperinvariant subspace for G^*, and we are done. Otherwise, since $Y \hookrightarrow X^*$, we have

$$X \subseteq X^{**} \hookrightarrow W,$$

and $G = B|_X$, thus, we may apply Corollary 2.7. Note that Z and W are nontrivial because $x \in Z$ and $x^* \in Y$. By Theorem 3.6(4), B satisfies the commuting condition for G, thus we obtain a hyperinvariant subspace from Corollary 2.7. ∎

Remark. Theorem 4.3(1), for $G \in B(X)$, appears in Theorem 1.1 in [1], except that the growth condition there is $\|G^k x\| = O(|k|^m)$, as $k \to \pm\infty$.

We may replace complete orbits $\{G^k x\}_{k \in \mathbf{Z}}$ with one-sided orbits $\{G^n x\}_{n \in \mathbf{N}}$, using the following construction.

Lemma 4.4. *Suppose $G \in B(X)$, and $\vec{\alpha}$ is a one-sided Beurling sequence such that*

$$\|G^n\| = O(\alpha(n)) \ (n \in \mathbf{N}),$$

and for all nontrivial $x \in X$,

$$\overline{\lim}_{n\to\infty} \frac{1}{\alpha(n)} \|G^n x\| > 0.$$

Then there exists a Banach space V and $H \in B(V)$ such that

(1) *$X \hookrightarrow V$;*

(2) $G = H|_X$;
(3) $\|x\|_V = \overline{\lim}_{n\to\infty} \frac{1}{\alpha(n)} \|G^n x\|$;
(4) $\|H^n\| \le \overline{\alpha}(n) \equiv \overline{\lim}_{m\to\infty} \frac{\alpha(m+n)}{\alpha(m)}$, *for all* $n \in \mathbf{N}$;
(5) $\|Hx\|_V \ge \|x\|_V$, *for all* $x \in V$;
(6) *X is dense in V*;
(7) *H satisfies the commuting condition for G.*

Proof. The construction of V and H satisfying (1)–(6) is in [18] (this is, the discrete analog of Lemma 3 in [18]). For (7), suppose $R \in B(X)$ and $RG = GR$. It is clear from (3) that $\|Rx\|_V \le \|R\|_{B(X)}\|x\|_V$, for all $x \in X$, thus, by (6), R extends uniquely to $S \in B(V)$ such that $\|S\|_{B(V)} \le \|R\|_{B(X)}$.

Remark. Note that $\overline{\alpha}(n)$ is much smaller, in general, then $\alpha(n)$. For example, if $\alpha(n) = n^m$, for some $m > 0$, or e^{n^r}, for $0 \le r < 1$, then $\overline{\alpha}(n) = 1$, for all $n \in \mathbf{N}$.

As an immediate corollary of Lemma 4.4 and Theorem 2.5, we have the following.

Theorem 4.5. *Suppose $G \in B(X)$, there exists a one-sided Beurling sequence $\vec{\alpha}$, and $x \in X$, such that*

$$\|G^n\| = O(\alpha(n)) \quad (n \in \mathbf{N})$$

and

$$\overline{\lim}_{n\to\infty} \frac{1}{\alpha(n)} \|G^n x\| > 0, \tag{4}$$

and there exist disjoint closed subsets Ω_j, $j = 1, 2$, of the complex plane, such that $X_G(\Omega_j) \neq \{\vec{0}\}$, for $j = 1, 2$. Then G has a pair of nontrivial closed hyperinvariant subspaces with trivial intersection.

Proof. We may assume that Eq. (4) is valid for all nontrivial x, otherwise,

$$\{x \mid \lim_{n\to\infty} \frac{1}{\alpha(n)} \|G^n x\| = 0\}$$

would be a nontrivial closed hyperinvariant subspace. Thus, we may apply Lemma 4.4 and Theorem 2.5. ■

Theorem 4.6. *Suppose $G \in B(X)$. There exists a one-sided Beurling sequence $\vec{\alpha}_1$, and $x \in X$, such that*

$$\|G^n\| = O(\alpha_1(n)) \quad (n \in \mathbf{N})$$

and

$$\overline{\lim}_{n\to\infty} \frac{1}{\alpha_1(n)} \|G^n x\| > 0, \tag{5}$$

and nontrivial $y \in C^\infty(G^{-1})$ such that

$$\|G^k y\| = O(\alpha_2(k)) \quad (k \in Z)$$

for some Beurling sequence $\vec{\alpha}_2$ and either

(1)

$$\overline{\alpha_1}(k) \equiv \overline{\lim}_{n\to\infty} \frac{\alpha_1(k+n)}{\alpha_1(n)} = e^{o(\sqrt{k})} \quad (k \in \mathbf{N}),$$

or

(2) *$\sigma(G, y)$ contains at least two points.*

Then G has a nontrivial closed hyperinvariant subspace.

Proof. As in the proof of Theorem 4.5, we may assume Eq. (5) is valid for all nontrivial x.

First, assume we are under hypothesis (1). Let V and H be as in Lemma 4.4. Since we are assuming $\mathrm{Im}(G)$ is dense in X, and by Lemma 4.4, X is dense in V and $X \hookrightarrow V$, it follows that $\mathrm{Im}(H)$ is dense in V. By Lemma 4.4(5), it now follows that H is invertible, and $\|H^{-1}\| \leq 1$. Thus, by Lemma 4.4(4), $\|H^k\| = O(\alpha_3(k))$, for $k \in \mathbf{Z}$, where

$$\alpha_3(k) \equiv \overline{\alpha_1}(k), \quad \alpha_3(-k) \equiv 1 \ \ \forall k \in \mathbf{N} \cup \{0\}.$$

As in the proof of Theorem 4.3, we have a Banach space

$$Y \hookrightarrow X$$

such that $\|(G|_Y)^k\| = O(\alpha_2(k))$, for $k \in \mathbf{Z}$. In Proposition 2.8, let $W \equiv Y^*$, $B \equiv (G|_Y)^*$, and let $Z \equiv V^*$, so that $G^*|_Z = H^*$. Then, as in the proof of Theorem 4.3,

$$Z \hookrightarrow X^* \hookrightarrow W,$$

$G^* = B|_{X^*}$, $\|(G^*|_Z)^k\| = \|H^k\| = O(\alpha_3(k))$, for $k \in \mathbf{Z}$, and B is decomposable. By Proposition 2.8, we now have a nontrivial closed hyperinvariant subspace for $(G^*)^{-1}$ and hence for G.

Under hypothesis (2), we construct Y, V and H as we did under hypothesis (1). By Lemma 4.4, $\|H^k\| = O(\overline{\alpha_1}(|k|))$, for $k \in \mathbf{Z}$. Note that $\vec{\overline{\alpha_1}}$ is a one-sided Beurling sequence. Thus, since $y \in Y$, we may invoke Corollary 2.7, with Z replaced by Y, W by V, and B by H. ■

A corollary is the following result, from Theorem 1.6 in [1] and [2].

Corollary 4.7. *Suppose G is a contraction. There exists a Beurling sequence $\vec{\alpha}$ and nontrivial $y \in C^\infty(G^{-1})$ such that $\|G^k y\| = O(\alpha(k))$, for $k \in \mathbf{Z}$, and there exists x such that*

$$\lim_{k\to\infty} \|G^k x\| \neq 0.$$

Then G has a nontrivial closed hyperinvariant subspace.

Remark. Corollary 4.7 may also be proven by using [17] to produce $x^* \in X^*$ such that $\{\|(G^*)^k x^*\|\}_{k\in\mathbf{Z}}$ is bounded, so that we may apply Theorem 4.3 to G^*.

Theorem 4.8. *Suppose $G \in B(X)$, $\vec{\alpha}$ is a one-sided Beurling sequence such that*

$$\|G^n\| = O(\alpha(n)) \quad (n \in \mathbf{N}),$$

and

$$\overline{\alpha}(k) \equiv \overline{\lim}_{n\to\infty} \frac{\alpha(k+n)}{\alpha(n)} = e^{o(\sqrt{k})} \quad (k \in \mathbf{N}),$$

and there exists $x \in X$, $x^* \in X^*$ such that

$$\overline{\lim}_{n\to\infty} \frac{1}{\alpha(n)} \|G^n x\| > 0, \quad \text{and} \quad \overline{\lim}_{n\to\infty} \frac{1}{\alpha(n)} \|(G^*)^n x^*\| > 0. \tag{6}$$

Then G has a nontrivial closed hyperinvariant subspace.

Proof. As in the first part of the proof of Theorem 4.6, use Lemma 4.4, this time on both G and G^*, to produce Banach spaces V, W such that

$$X \hookrightarrow V, \quad X^* \hookrightarrow W,$$

and operators $H_1 \in B(V), H_2 \in B(W)$ such that $G = H_1|_X, G^* = H_2|_{X^*}$, $\|H_j^k\| = O(\alpha_3(k))$, for $k \in \mathbf{Z}$, $j = 1, 2$, where $\vec{\alpha_3}$ is defined in the proof of Theorem 4.6.

As in the proof of Theorem 4.6, let $Z \equiv V^*$ so that $G^*|_Z = H_1^*$, and invoke Proposition 2.8 to produce a nontrivial closed invariant subspace for G^*. Note that Lemma 4.4(7) implies that it is hyperinvariant. ■

Remark. When X is reflexive, this result appears in Theorem 5.1.9 in [4] and Theorem 1.4 in [1], except that there $\overline{\alpha}(k)$ is $O(k^m)$ for some $m > 0$.

Corollary 4.9. *Suppose G is a contraction, and there exist $x \in X$, $x^* \in X^*$ such that*

$$\lim_{n\to\infty} \|G^n x\| \neq 0, \quad \lim_{n\to\infty} \|(G^*)^n x^*\| \neq 0.$$

Then G has a nontrivial closed hyperinvariant subspace.

In fact, for operators satisfying (3.4) for all nontrivial $x \in X$, $x^* \in X^*$, we may construct a large family of hyperinvariant subspaces for G^*.

Definition 4.10. [4, Chpt. 5.2] *If $\vec{\alpha}$ is a weight sequence, define $\mathcal{U}[\vec{\alpha}]$ to be the Banach algebra of functions in L^1 of the unit disc, whose Fourier coefficients $\{\hat{f}(k)\}_{k\in\mathbf{Z}}$ satisfy the growth condition*

$$\|f\|_{\mathcal{U}[\vec{\alpha}]} \equiv \sum_{k\in\mathbf{Z}} |\hat{f}(k)|\alpha(k) < \infty.$$

Theorem 4.11. *Suppose $\vec{\alpha}$ is a one-sided Beurling sequence.*

$$\|G^k\| = O(\alpha(k)),$$

$$\overline{\alpha}(k) \equiv \overline{\lim}_{n\to\infty} \frac{\alpha(k+n)}{\alpha(n)} = e^{o(\sqrt{k})} \quad (k \in \mathbf{N}),$$

and

$$\overline{\lim}_{n\to\infty} \frac{1}{\alpha(n)} \|G^n x\| > 0, \quad \textit{and} \quad \overline{\lim}_{n\to\infty} \frac{1}{\alpha(n)} \|(G^*)^n x^*\| > 0,$$

for all nontrivial $x \in X$, $x^* \in X^*$.

For Ω, a closed subset of the unit circle Υ, define $E(\Omega)$ to be the set of all $x \in X$ such that

$$\lim_{N\to\infty} \overline{\lim}_{n\to\infty} \frac{1}{\alpha(n)} \| \sum_{k=-N}^{N} \hat{f}(k)(G^*)^{k+n} x\| = 0,$$

whenever $f \in \mathcal{U}[\vec{\alpha_3}]$ has support disjoint from Ω, where

$$\alpha_3(k) \equiv \overline{\alpha_1}(k), \ \ \alpha_3(-k) \equiv 1 \ \forall k \in \mathbf{N} \cup \{0\}.$$

Then $E(\Omega)$ is as in Proposition 2.4, with G replaced by G^*, and contains $F(\Omega)$, defined to be the set of all $x \in C^\infty((G^*)^{-1})$ such that

$$\lim_{N,M\to\infty} \sup_{k\in\mathbf{Z}} \frac{1}{\alpha_3(k)} \| \sum_{j=-N}^{M} \hat{f}(j)(G^*)^{k+j} x\| = 0,$$

whenever $f \in \mathcal{U}[\vec{\alpha_3}]$ has support disjoint from Ω.

We have

$$F(\Omega) \subseteq X_G(\Omega) \subseteq E(\Omega).$$

In particular, $E(\Omega)$ and $E(\Upsilon - \Omega)$ are a pair of nontrivial closed hyperinvariant subspace for G^*, with trivial intersection, whenever both $F(\Omega)$ and $F(\Upsilon - \Omega)$ are nontrivial.

Proof. By the proof of Theorem 4.8, we have Banach spaces Z, W, $B \in B(W)$, such that

$$Z \hookrightarrow X^* \hookrightarrow W,$$

$G^* = B|_{X^*}$, and both $\|B^k\|$ and $\|(G^*|_Z)^k\|$ are $O(\alpha_3(k))$, for $k \in \mathbf{Z}$, with

$$\|x\|_W \equiv \overline{\lim}_{n\to\infty} \frac{1}{\alpha(n)} \|(G^*)^n x\|.$$

By Theorem 3.6(3), we may assume $Z = Z_{\vec{\alpha_3}}(G^*)$, so that

$$\|x\|_Z = \sup_{k\in\mathbf{Z}} \frac{1}{\alpha_3(k)} \|G^k x\|.$$

The growth condition on the powers of B and $(G^*)|_Z$ imply that these operators are $\mathcal{U}([\vec{\alpha_3}])$-unitary [4, Chpt. 5], with functional calculus

$$f(B) = \sum_{k=-\infty}^{\infty} \hat{f}(k) B^k \quad (f \in \mathcal{U}[\vec{\alpha_3}])$$

and local spectral subspaces

$$W_B(\Omega) = \bigcap \{\mathcal{N}(f(B)) \mid f \in \mathcal{U}[\vec{\alpha_3}], \text{ support of } f \text{ is disjoint from } \Omega\}$$

[4, Chpt. 3.1]. Thus, $Z_{(G^*|_Z)}(\Omega) = F(\Omega)$ and $E(\Omega)$, as in the statement of this theorem, equals $X \cap W_B(\Omega)$ as in Proposition 2.4. ∎

Definition 4.12. *A contraction G is a $C_{1,1}$ contraction if*

$$\lim_{n\to\infty} \|G^n x\| \neq 0 \quad and \quad \lim_{n\to\infty} \|(G^*)^n x^*\| \neq 0,$$

for all nontrivial $x \in X, x^* \in X^*$.

Putting $\alpha(k) \equiv 1$ in Theorem 4.11 gives us the following:

Corollary 4.13. *Suppose G is a $C_{1,1}$ contraction. For Ω, a closed subset of the unit circle Υ, define $E(\Omega)$ to be the set of all $x \in X$ such that*

$$\lim_{N\to\infty} \overline{\lim}_{n\to\infty} \| \sum_{k=-N}^{N} \hat{f}(k)(G^*)^{k+n} x \| = 0,$$

whenever $f \in C^\infty(\Upsilon)$ has support disjoint from Ω. Then $E(\Omega)$ is as in Proposition 2.4, with G replaced by G^*, and contains $F(\Omega)$, defined to be the set of all $x \in C^\infty(G^{-1})$ such that

$$\lim_{N,M\to\infty} \sup_{k\in\mathbf{Z}} \| \sum_{j=-N}^{M} \hat{f}(j)(G^*)^{k+j} x \| = 0,$$

whenever $f \in C^\infty(\Upsilon)$ has support disjoint from Ω.

We have

$$F(\Omega) \subseteq X_G(\Omega) \subseteq E(\Omega).$$

In particular, $E(\Omega)$ and $E(\Upsilon - \Omega)$ are a pair of nontrivial closed hyperinvariant subspaces for G^*, with trivial intersection, whenever both $F(\Omega)$ and $F(\Upsilon - \Omega)$ are nontrivial.

References

1. A. Atzmon, On the existence of hyperinvariant subspaces, *J. Operator Theory* **11** (1984) 3–40.
2. B. Beauzamy, Sous-espaces invariants de type fonctionnel dans les espaces de Banach, *Acta. Math.* **144** (1980) 65–82.
3. B. Beauzamy, *Introduction to Operator Theory and Invariant Subspaces*, North-Holland, Amsterdam-New York-Oxford-Tokyo, 1988.
4. I. Colojoara and C. Foias, *Theory of Generalized Spectral Operators*, Gordon and Breach, New York, 1968.
5. R. deLaubenfels, *Existence Families, Functional Calculi and Evolution Equations*, Lecture Notes in Mathematics, Vol. 1570, Springer-Verlag, 1994.
6. R. deLaubenfels and S. Kantorovitz, Laplace and Laplace–Stieltjes spaces, *J. Func. An.* **116** (1993) 1–61.
7. R. deLaubenfels and Vu Quoc Phong, Decomposable embeddings, complete trajectories, and invariant subspaces, *Studia Math.* **119** (1996) 65–76.
8. R. deLaubenfels and Vu Quoc Phong, The discrete Hille–Yosida space and the asymptotic behaviour of individual orbits of linear operators, *J. Func. An.* **142** (1996) 539–548.
9. I. Erdelyi and S. Wang, *A Local Spectral Theory for Closed Operators*, London Mathematical Society Lecture Notes, Vol. 105, London Mathematical Society, Cambridge, 1985.
10. S. Huang, Characterizing spectra of closed operators through existence of slowly growing solutions of their Cauchy problems, *Studia Math.* **116** (1995) 23–41.

11. S. Kantorovitz, *Spectral Theory of Banach Space Operators*, Lecture Notes in Mathematics, Vol. 1012, Springer-Verlag, 1983.
12. S. Kantorovitz, The Hille–Yosida space of an arbitrary operator, *J. Math. An. Appl.* **136** (1988) 107–111.
13. S.G. Krein, G.I. Laptev, and G. A. Cretkova, On Hadamard correctness of the Cauchy problem for the equation of evolution, *Soviet Math. Dokl.* **11** (1970) 763–766.
14. R. Lange and S. Wang, *New Approaches in Spectral Decomposition*, American Mathematical Society Contemporary Mathematics, Vol. 128, American Mathematical Society, Providence, 1992.
15. M.M. Neumann, Local spectral theory for operators on Banach spaces and applications to convolution operators on group algebras, *Louisiana State University Seminar Notes in Functional Analysis and PDEs*, 1993–1994, pp. 287–308.
16. F.H. Vasilescu, *Analytic Functional Calculus and Spectral Decompositions*, D. Reidel, Buchurest-Dordrecht, 1982.
17. Vu Quoc Phong, On the spectrum, complete trajectories, and asymptotic stability of linear semi-dynamical systems, *J. Diff. Eq.* **105** (1993) 30–45.
18. Vu Quoc Phong, Semigroups with non-quasi-analytic growth, *Studia Math.* **104** (1993) 229–241.