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Clifford Semilattice Decompositions of Stratified Normal Bands of Monoids

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Abstract. In this paper, we give a twisted spined product structure of a stratified band of monoids and prove that a stratified normal band of monoids is isomorphic to a Clifford semilattice of Rees matrix semigroups over some of these monoids.

1. Introduction

Petrich has proved that a semigroup which is a normal band of groups is a Clifford semilattice of completely simple semigroups [5, Construction 4.2; 6, Corollary 6.3; 7, Theorem IV.2.3]. A semigroup is completely simple if and only if it is isomorphic to a regular Rees matrix semigroup $\mathcal{U}(G, I, \Lambda; P)$ over a group G [1, Subsec. 3.5]. As a natural way of generalizing the concept of a completely simple semigroup, Lallement [3, Definition 3.4] and Petrich [7, Definition III. 2.10] have introduced a Rees matrix semigroup $\mathcal{U}(M, I, \Lambda; P)$ over a monoid. Now, a problem arises: If S is a normal band of monoids, is S a Clifford semilattice of Rees matrix semigroups over monoids?

Schein has proved that an E-band of monoids is proper if and only if it is isomorphic to a spined product of a Clifford semilattice of some of these monoids and E [2, Theorem 1]. In this paper, we introduce a twisted spined product structure of an arbitrary stratified band of monoids. As an application of this structure theorem, we prove that a stratified normal band of monoids is isomorphic to a Clifford semilattice of Rees matrix semigroups over some of these monoids.

In this paper, the symbol B^A denotes the set of all functions from a set A into a set B; ι_S denotes the identity automorphism of a semigroup S; $S = [Y; S_\alpha, \sigma_{\alpha,\beta}]$ denotes a Clifford (strong) semilattice Y of semigroups $\{S_\alpha\}_{\alpha \in Y}$ with respect to a system of transitive homomorphisms $\Sigma = \{\sigma_{\alpha,\beta} \mid \alpha, \beta \in Y, \alpha \geq \beta\}$ [2; 7, I.8.7].

A band E is an *idempotent semigroup*. A band E is a *rectangular band* if $iji = i$ for all $i, j \in E$. It is *normal* if and only if it is isomorphic to a Clifford semilattice of rectangular bands (cf. [4, Proposition 5.14]). A semigroup S is an *E-band of a family of semigroups* $\{S_i\}_{i\in E}$ if $\{S_i\}_{i\in E}$ is a partion of S into classes of a congruence relation, i.e., for any $i, j \in E$, $S_iS_j \subseteq S_{ij}$. A *monoid M* is a semigroup M with an identity 1. An E-band S of a family $\mathcal{S}$ of monoids $\{M_i\}_{i\in E}$ is called *stratified* if $1_i1_j = 1_j$ for all $i, j \in E$ such that $ij = j$ and $1_j1_i = 1_j$ for all $i, j \in E$ such that $ji = j$. An E-band S is called *proper* if $1_i1_j = 1_{ij}$ for all $i, j \in E$ (see [2, Sec. 1, Definition 1]).

Lemma 1. [2, Lemma 1] *An E-band S of a family $\mathcal{S}$ of monoids $\{M_i\}_{i\in E}$ is stratified if and only if $1_i1_{ij} = 1_{ij}$ and $1_{ij}1_j = 1_{ij}$ for all $i, j \in E$. Every proper band of monoids is stratified.*

Lemma 2. [2, Corollary 10] *Given a monoid M, a rectangular band E, and a family $P = (p_i)_{i\in E}$ of invertible elements of M, consider a multiplication on the set $M \times E$:*

$$(s, i)(t, j) = (sp_{ji}t, ij).$$

Then $M \times E$ is a stratified E-band of monoids isomorphic to M and every stratified E-band of monoids can be so obtained

In this paper, we denote the semigroup constructed in Lemma 2 by $S = \mathcal{U}(M, E; P)$. It is clear that $S = \mathcal{U}(M, E; P)$ is isomorphic to a Rees matrix semigroup over the monoid M (cf. [3, Definition 3.4] or [7, Definition III.2.10]).

Construction. Let Y be a semilattice and let E be a band which is the semilattice Y of a family of rectangular bands $\{E_\alpha\}_{\alpha\in Y}$. To each $\alpha \in Y$, we associate a monoid M_α and a family $P_\alpha = (p_i)_{i\in E_\alpha}$ of invertible elements of M_α, and suppose $M_\alpha \cap M_\beta = \emptyset$ if $\alpha \neq \beta$. For each pair $\alpha, \beta \in Y$, $\alpha \geq \beta$, let $\omega_{\alpha,\beta} : M_\alpha \times E_\alpha \to M_\beta^{E_\beta}$ be a function, with $\omega_{\alpha,\beta} : a \mapsto \omega^a_{\alpha,\beta} (\forall a \in M_\alpha \times E_\alpha)$, satisfying the following conditions: For arbitrary $\alpha, \beta \in Y$ and $a = (s, i) \in M_\alpha \times E_\alpha$, $b = (t, j) \in M_\beta \times E_\beta$,

(i) if $\alpha \geq \beta$, then $\omega^a_{\alpha,\beta}(j'j) = \omega^a_{\alpha,\beta}(j)$ for all $j', j \in E_\beta$;

(ii) $\omega^a_{\alpha,\alpha}(i') = p_{ii'}s$ for all $i' \in E_\alpha$;

on $S = \cup_{\alpha\in Y}(M_\alpha \times E_\alpha)$ we define a multiplication $*$ by

$$a * b = (p^{-1}_{ij}\,\omega^a_{\alpha,\alpha\beta}(ij)\omega^b_{\beta,\alpha\beta}(ji), ij). \tag{1}$$

(iii) if $\gamma \leq \alpha\beta$, then $\omega^{a*b}_{\alpha\beta,\gamma}(r) = \omega^a_{\alpha,\gamma}(r)\omega^b_{\beta,\gamma}(ri)$ for all $r \in E_\gamma$;

(iv) for $e_i = (p^{-1}_i, i)$, $\omega^{e_i}_{\alpha,\alpha\beta}(ij) = p_{ij}p^{-1}_{iji}$ and $\omega^{e_i}_{\alpha,\alpha\beta}(ji) = 1_{\alpha\beta}$.

The resulting system $(S, *)$ is called the *twisted spined product* of the disjoint union of monoids $M = \cup_{\alpha\in Y}M_\alpha$ and the band $E = \cup_{\alpha\in Y}E_\alpha$, with respect to $P = \{P_\alpha\}_{\alpha\in Y}$ and $\omega = \{\omega_{\alpha,\beta} \mid \alpha, \beta \in Y, \alpha \geq \beta\}$, over the common semilattice skeleton Y. We denote this system by $S = M \otimes_{Y,P,\omega} E$.

Theorem 1. *Given a semilattice Y, a family $\mathcal{M}$ of monoids $\{M_\alpha\}_{\alpha\in Y}$, and a band $E = \cup_{\alpha\in Y}E_\alpha$, the system $S = M \otimes_{Y,P,\omega} E$ obtained in the construction is a semigroup and a stratified E-band of a family of monoids $\{M_\alpha \times \{i\}\}_{i\in E_\alpha,\alpha\in Y}$.*

Every stratified E-band of a family of monoids is isomorphic to a twisted spined product of a disjoint union of some of these monoids and E.

Proof. Direct part. For arbitrary elements $a = (s, i) \in M_\alpha \times E_\alpha$, $b = (t, j) \in M_\beta \times E_\beta$, $c = (u, k) \in M_\gamma \times E_\gamma$, let $(v, l) = (a * b) * c$. In view of (i), (iii), and formula (1), we have $l = ijk$ and

$$\begin{aligned} v &= p_{(ij)k}^{-1}\omega^{a*b}_{\alpha\beta,\alpha\beta\gamma}[(ij)k]\omega^{c}_{\gamma,\alpha\beta\gamma}[k(ij)] \\ &= p_{ijk}^{-1}\omega^{a}_{\alpha,\alpha\beta\gamma}(ijk)\omega^{b}_{\beta,\alpha\beta\gamma}[(ijk)i]\omega^{c}_{\gamma,\alpha\beta\gamma}(kij) \\ &= p_{ijk}^{-1}\omega^{a}_{\alpha,\alpha\beta\gamma}(ijk)\omega^{b}_{\beta,\alpha\beta\gamma}(jki)\omega^{c}_{\gamma,\alpha\beta\gamma}(kij) \end{aligned}$$

since $ijki = (ijk)(jki)$. Analogously, we have $a * (b * c) = (v, l)$. Thus $(S, *)$ is a semigroup.

It is obvious that a mapping $\pi \colon S \to E$ defined by

$$\pi(s, i) = i, \ \ \forall (s, i) \in M_\alpha \times E_\alpha, \alpha \in Y,$$

is a homomorphism of S onto E. For every $i \in E_\alpha$, $\alpha \in Y$, we have

$$\pi^{-1}(i) = M_\alpha \times \{i\}$$

on which the multiplication $*$, by (ii) and formula (1), is given as follows: For every $s, t \in M_\alpha$,

$$\begin{aligned} (s, i) * (t, i) &= (p_{ii}^{-1}\omega^{(s,i)}_{\alpha,\alpha\alpha}(ii)\omega^{(t,i)}_{\alpha,\alpha\alpha}(ii), ii) \\ &= (p_i^{-1}p_i s p_i t, i) = (s p_i t, i). \end{aligned}$$

Thus, $(M_\alpha \times \{i\}, *)$ is a monoid with the identity $e_i = (p_i^{-1}, i)$. Hence, S is an E-band of a family of monoids $\mathcal{S} = \{M_\alpha \times \{i\}\}_{i \in E_\alpha, \alpha \in Y}$.

For arbitrary $\alpha, \beta \in Y$ and $i \in E_\alpha$, $j \in E_\beta$, in view of (i), (ii), (iv), and formula (1), we have

$$\begin{aligned} e_i * e_{ij} &= (p_{i(ij)}^{-1}\omega^{e_i}_{\alpha,\alpha\beta}[i(ij)]\omega^{e_{ij}}_{\alpha\beta,\alpha\beta}[(ij)i], i(ij)) \\ &= (p_{ij}^{-1}p_{ij}p_{iji}^{-1}p_{ij(ij)i}p_{ij}^{-1}, ij) \\ &= (p_{ij}^{-1}, ij) = e_{ij}. \end{aligned}$$

Analogously, we have $e_{ij} * e_j = e_{ij}$. From these, it follows by Lemma 1 that S is a stratified E-band of a family $\mathcal{S}$ of monoids $\{\pi^{-1}(i)\}_{i \in E}$.

Converse part. Let S be a stratified E-band of a family $\mathcal{S}$ of monoids $\{M_i\}_{i \in E}$. Then there exist a homomorphism f of S onto E and a homomorphism g of E onto a semilattice Y such that

$$(\forall i \in E) \ \ f^{-1}(i) = M_i,$$
$$(\forall \alpha \in Y) \ \ g^{-1}(\alpha) = E_\alpha,$$

where E_α is a rectangular band for every $\alpha \in Y$, respectively. Let $\delta = gf$. Then δ is a homomorphism of S onto Y such that

$$(\forall \alpha \in Y) \ \ \delta^{-1}(\alpha) = f^{-1}(g^{-1}(\alpha)) = \cup_{i \in E_\alpha} M_i.$$

Let $S_\alpha = \delta^{-1}(\alpha)$. Then S_α is a stratified rectangular E_α-band of a family of monoids $\{M_i\}_{i\in E_\alpha}$ and S is the semilattice Y of subsemigroups $\{S_\alpha\}_{\alpha\in Y}$.

In view of Lemma 2, to each $\alpha \in Y$, there exists a monoid which is one member of the family of $\{M_i\}_{i\in E_\alpha}$, say M_α, and a family P_α of invertible elements $(p_i)_{i\in E_\alpha}$ of M_α such that $S_\alpha \cong \mathcal{U}(M_\alpha, E_\alpha; P_\alpha)$. We may identify S_α with $\mathcal{U}(M_\alpha, E_\alpha; P_\alpha)$ and assume that $M_\alpha \cap M_\beta = \emptyset$ if $\alpha \neq \beta$. From this, a simple argument shows that $M_i = M_\alpha \times \{i\}$ for all $i \in E_\alpha$ and $\alpha \in Y$, so that

$$(\forall \alpha \in Y)(\forall (s,i) \in S_\alpha) \quad f(s,i) = i. \tag{2}$$

We fix $\alpha, \beta \in Y$ such that $\alpha \geq \beta$ and let $a = (s,i) \in S_\alpha$. If $j \in E_\beta$, then $(1_\beta, j)a = (\omega^a_{\alpha,\beta}(j), j_1) \in S_\beta$. Since f is a homomorphism, by formula (2), we have

$$j_1 = f((1_\beta, j)a) = f(1_\beta, j)f(a) = ji.$$

Hence, there exists a function $\omega_{\alpha,\beta} : M_\alpha \times E_\alpha \to M_\beta^{E_\beta}$ defined by

$$\omega_{\alpha,\beta} : a \mapsto \omega^a_{\alpha,\beta}$$

with

$$\omega^a_{\alpha,\beta} : j \mapsto \omega^a_{\alpha,\beta}(j) \ \ (\forall j \in E_\beta)$$

such that

$$(1_\beta, j)a = (\omega^a_{\alpha,\beta}(j), ji), \ \ a \in S_\alpha, j \in E_\beta, \ \ (\alpha \geq \beta). \tag{3}$$

For any $\alpha, \beta \in Y$, $a = (s,i) \in S_\alpha$, $b = (t,j) \in S_\beta$, let $(q,k) = ab \in S_{\alpha\beta}$. Then $k = f(ab) = f(a)f(b) = ij$ by formula (2). Using formula (3), we have

(i) If $\alpha \geq \beta$, for every $j', j \in E_\beta$, from

$$\begin{aligned}(\omega^a_{\alpha,\beta}(j'j), (j'j)i)a &= (1_\beta, j'j)a = (p^{-1}_{jj'}, j')(1_\beta, j)a = (p^{-1}_{jj'}, j')(\omega^a_{\alpha,\beta}(j), ji) \\ &= (p^{-1}_{j(ji)j'} p_{(ji)j'}(\omega^a_{\alpha,\beta}(j)), j'(ji)) = (\omega^a_{\alpha,\beta}(j), j'ji),\end{aligned}$$

we obtain $\omega^a_{\alpha,\beta}(j'j) = \omega^a_{\alpha,\beta}(j)$.

(ii) For every $i' \in E_\alpha$, from

$$(\omega^a_{\alpha,\alpha}(i'), i'i) = (1_\alpha, i')a = (1_\alpha, i')(s,i) = (p_{ii'}s, i'i),$$

we obtain $\omega^a_{\alpha,\alpha}(i') = p_{ii'}s$.

(iii) If $\gamma \leq \alpha\beta$, for every $r \in E_\gamma$, from

$$\begin{aligned}(\omega^{ab}_{\alpha\beta,\gamma}(r), r(ij)) &= (1_\gamma, r)ab = (\omega^a_{\alpha,\gamma}(r), ri)b \\ &= ((\omega^a_{\alpha,\gamma}(r))p^{-1}_{(ri)r}, r)(1_\gamma, ri)b \\ &= ((\omega^a_{\alpha,\gamma}(r))p^{-1}_{r(ri)r}, r)(\omega^b_{\beta,\gamma}(ri), (ri)j) \\ &= ((\omega^a_{\alpha,\gamma}(r))p_r^{-1} p_{(rij)r}[\omega^b_{\beta,\gamma}(ri)], r(rij)) \\ &= (\omega^a_{\alpha,\gamma}(r)\omega^b_{\beta,\gamma}(ri), rij),\end{aligned}$$

we obtain $\omega^{ab}_{\alpha\beta,\gamma}(r) = \omega^{a}_{\alpha,\gamma}(r)\omega^{b}_{\beta,\gamma}(ri)$. From this, it follows by (i) and (ii) that

$$\begin{aligned} p_{ij}q = p_{(ij)(ij)}q &= \omega^{ab}_{\alpha\beta,\alpha\beta}(ij) = \omega^{a}_{\alpha,\alpha\beta}(ij)\omega^{b}_{\beta,\alpha\beta}[(ij)i] \\ &= \omega^{a}_{\alpha,\alpha\beta}(ij)\omega^{b}_{\beta,\alpha\beta}[(ij)(ji)] = \omega^{a}_{\alpha,\alpha\beta}(ij)\omega^{b}_{\beta,\alpha\beta}(ji). \end{aligned}$$

Thus, $q = p_{ij}^{-1}\omega^{a}_{\alpha,\alpha\beta}(ij)\omega^{b}_{\beta,\alpha\beta}(ji)$ and

$$ab = (q, ij) = a * b, \ \ \forall a, b \in S. \tag{4}$$

(iv) Since S is a stratified E-band of monoids $M_\alpha \times \{i\}$ with identities $e_i = (p_i^{-1}, i)$ $(i \in E_\alpha, \alpha \in Y)$, by (ii), (4), and Lemma 1, we have

$$\begin{aligned} (p_{ij}^{-1}, ij) = e_{ij} = e_i e_{ij} &= (p_i^{-1}, i)(p_{ij}^{-1}, ij) \\ &= (p_{i(ij)}^{-1}\omega^{e_i}_{\alpha,\alpha\beta}[i(ij)]\omega^{e_{ij}}_{\alpha\beta,\alpha\beta}[(ij)i], i(ij)) \\ &= (p_{ij}^{-1}\omega^{e_i}_{\alpha,\alpha\beta}(ij)p_{(ij)(iji)}p_{ij}^{-1}, ij), \end{aligned}$$

which shows that $\omega^{e_i}_{\alpha,\alpha\beta}(ij) = p_{ij}p_{iji}^{-1}$. Similarly, by (i), (ii), and (4), it follows from $e_{ji} = e_{ji}e_i$ that $\omega^{e_i}_{\alpha,\alpha\beta} = 1_{\alpha\beta}$.

As stated above, we conclude that $S = M \otimes_{Y,P,\omega} E$. ■

Theorem 2. *If S is a stratified normal band of a family of monoids, then S is isomorphic to a Clifford semilattice of a family of Rees matrix semigroups over some of these monoids, i.e., S is isomorphic to a Clifford semilattice of a family of stratified rectangular bands of some of these monoids.*

Proof. Let $E = [Y; E_\alpha, \theta_{\alpha,\beta}]$ be a normal band which is a Clifford semilattice Y of a family of rectangular bands $\{E_\alpha\}_{\alpha\in Y}$. Let S be a stratified E-band of a family S of monoids $\{M_i\}_{i\in E}$. In view of Theorem 1 and its proof, we have $S = M \otimes_{Y,P,\omega} E$ and S is a semilattice Y of $S_\alpha = \mathcal{U}(M_\alpha, E_\alpha; P_\alpha)$ which is isomorphic to a stratified E_α-band of a family of monoids $\{M_i\}_{i\in E_\alpha}$ $(\alpha \in Y)$. First, we claim that

(v) for every $\alpha, \beta \in Y, a = (s, i) \in S_\alpha, b = (t, j) \in S_\beta$, and for all $k \in E_{\alpha\beta}$, we have

$$p_{ijk}^{-1}\omega^{a}_{\alpha,\alpha\beta}(k)\omega^{b}_{\beta,\alpha\beta}(ki) = p_{ij}^{-1}\omega^{a}_{\alpha,\alpha\beta}(ij)\omega^{b}_{\beta,\alpha\beta}(ji).$$

In fact, let $(q, ij) = a * b = (p_{ij}^{-1}\omega^{a}_{\alpha,\alpha\beta}(ij)\omega^{b}_{\beta,\alpha\beta}(ji), ij)$. For any $k \in E_{\alpha\beta}$, $e_k = (p_k^{-1}, k)$, by (i) and (ii), we have

$$\begin{aligned} \omega^{e_k}_{\alpha\beta,\alpha\beta}(kij) &= \omega^{e_k}_{\alpha\beta,\alpha\beta}(ij) = p_{k(ij)}p_k^{-1}, \\ \omega^{a*b}_{\alpha\beta,\alpha\beta}(ijk) &= \omega^{a*b}_{\alpha\beta,\alpha\beta}k. \end{aligned}$$

In view of formulas (1), (iii), and the equalities above, we have

$$\begin{aligned} (qp_{k(ij)}p_k^{-1}, ijk) = (q, ij) * (p_k^{-1}, k) &= (a * b) * e_k \\ &= (p_{ijk}^{-1}\omega^{a*b}_{\alpha\beta,\alpha\beta}(ijk)\omega^{e_k}_{\alpha\beta,\alpha\beta}(kij).ijk) \\ &= (p_{ijk}^{-1}\omega^{a}_{\alpha,\alpha\beta}(k)\omega^{b}_{\beta,\alpha\beta}(ki)p_{k(ij)}p_k^{-1}, ijk), \end{aligned}$$

which shows that $q = p_{ijk}^{-1}\omega_{\alpha,\alpha\beta}^a(k)\omega_{\beta,\alpha\beta}^b(ki)$, so (v) holds.

For any $\alpha, \beta \in Y, \alpha \geq \beta$, we define a mapping $\rho_{\alpha,\beta} : S_\alpha \to S_\beta$ by

$$\rho_{\alpha,\beta}(a) = (p_{\theta_{\alpha,\beta}i}^{-1}\omega_{\alpha,\beta}^a(\theta_{\alpha,\beta}i), \theta_{\alpha,\beta}i), \quad \forall a = (s,i) \in S_\alpha. \tag{5}$$

By putting $l = \theta_{\alpha,\beta}i \in E_\beta$ and $e_l = (p_l^{-1}, l)$, for any $k \in E_\beta$, with $E = [Y; E_\alpha, \theta_{\alpha,\beta}]$ and $\alpha\beta = \beta$, we have

$$lki = l(\theta_{\beta,\beta}k)(\theta_{\alpha,\beta}i) = lkl = l$$

and analogously, $li = il = l$. In view of (ii), we have

$$\omega_{\beta,\alpha\beta}^{e_l}(li) = \omega_{\beta,\beta}^{e_l}(l) = p_{ll}p_l^{-1} = 1_\beta,$$
$$\omega_{\beta,\alpha\beta}^{e_l}(ki) = p_{l(ki)}p_l^{-1} = 1_\beta.$$

By formulas (1), (5), (v), and the equalities above, we have

$$\begin{aligned}\rho_{\alpha,\beta}(a) &= (p_{il}^{-1}\omega_{\alpha,\alpha\beta}^a(il)\omega_{\beta,\alpha\beta}^{e_l}(li), l) = a * e_l \\ &= (p_{ilk}^{-1}\omega_{\alpha,\alpha\beta}^a(k)\omega_{\beta,\alpha\beta}^{e_l}(ki), l) \\ &= (p_{(\theta_{\alpha,\beta}i)k}^{-1}\omega_{\alpha,\beta}^a(k), \theta_{\alpha,\beta}i) \quad (\forall k \in E_\beta).\end{aligned} \tag{6}$$

Let $a = (s,i), b = (t,j) \in S_\alpha$. By (i), (iii), (1), (5), and (6), we have

$$\begin{aligned}\rho_{\alpha,\beta}(a) &= (p_{(\theta_{\alpha,\beta}i)(\theta_{\alpha,\beta}j)}^{-1}\omega_{\alpha,\beta}^a(\theta_{\alpha,\beta}j), \theta_{\alpha,\beta}i) \\ &= (p_{\theta_{\alpha,\beta}(ij)}^{-1}\omega_{\alpha,\beta}^a[\theta_{\alpha,\beta}(ij)], \theta_{\alpha,\beta}i)\end{aligned}$$

and $\rho_{\alpha,\beta}(b) = (p_{(\theta_{\alpha,\beta}j)(\theta_{\alpha,\beta}i)}^{-1}\omega_{\alpha,\beta}^b[\theta_{\alpha,\beta}(ji)], \theta_{\alpha,\beta}j)$, so

$$\begin{aligned}\rho_{\alpha,\beta}(a) * \rho_{\alpha,\beta}(b) &= (p_{\theta_{\alpha,\beta}}(ij)^{-1}\omega_{\alpha,\beta}^a[\theta_{\alpha,\beta}(ij)]\omega_{\alpha,\beta}^b[(\theta_{\alpha,\beta}(ij)i], (\theta_{\alpha,\beta}i)(\theta_{\alpha,\beta}j)) \\ &= (p_{\theta_{\alpha,\beta}(ij)}^{-1}\omega_{\alpha,\beta}^{a*b}[\theta_{\alpha,\beta}(ij)], \theta_{\alpha,\beta}(ij)) = \rho_{\alpha,\beta}(a*b).\end{aligned}$$

Thus, $\rho_{\alpha,\beta}$ is a homomorphism of S_α into S_β.

(A) For any $a = (s,i) \in S_\alpha$, since $\theta_{\alpha,\alpha}i = i$, by (ii) and (5), we have

$$\rho_{\alpha,\alpha}(a) = (p_i^{-1}\omega_{\alpha,\alpha}^a(i), i) = (p_i^{-1}p_{ii}s, i) = (s,i) = a.$$

Hence, $\rho_{\alpha,\alpha} = \iota_{S_\alpha}$ for all $\alpha \in Y$.

(B) Let $\alpha, \beta, \gamma \in Y$ be such that $\alpha \geq \beta \geq \gamma$. For every $a = (s,i) \in S_\alpha$, $e = (p_{\theta_{\alpha,\beta}i}^{-1}, \theta_{\alpha,\beta}i) \in S_\beta$. Since

$$\theta_{\beta,\gamma}\theta_{\alpha,\beta}i = \theta_{\alpha,\gamma}i, \quad (\theta_{\alpha,\gamma}i)i = \theta_{\alpha,\gamma}i,$$
$$(\theta_{\alpha,\beta}i)(\theta_{\alpha,\gamma}i) = \theta_{\alpha,\gamma}i = (\theta_{\alpha,\gamma}i)(\theta_{\alpha,\beta}i),$$

by (iv) and the equalities above, we have

$$\begin{aligned}\omega_{\beta,\gamma}^e[(\theta_{\alpha,\gamma}i)i] &= \omega_{\beta,\beta\gamma}^e[(\theta_{\alpha,\beta}i)(\theta_{\alpha,\gamma}i)] \\ &= p_{(\theta_{\alpha,\beta}i)(\theta_{\alpha,\gamma}i)}p_{(\theta_{\alpha,\beta}i)(\theta_{\alpha,\gamma}i)(\theta_{\alpha,\gamma}i)}^{-1} \\ &= p_{\theta_{\alpha,\gamma}i}p_{\theta_{\alpha,\gamma}i}^{-1} = 1_\gamma,\end{aligned}$$

so

$$\rho_{\beta,\gamma}\rho_{\alpha,\beta}(a) = \rho_{\beta,\gamma}(a * e) = (p^{-1}_{\theta_{\beta,\gamma}\theta_{\alpha,\beta}i}\omega^{a*e}_{\alpha\beta,\gamma}(\theta_{\beta,\gamma}\theta_{\alpha,\beta}i), \theta_{\beta,\gamma}\theta_{\alpha,\beta}i)$$
$$= (p^{-1}_{\theta_{\alpha,\gamma}i}\omega^{a}_{\alpha,\gamma}(\theta_{\alpha,\gamma}i)\omega^{e}_{\beta,\gamma}[(\theta_{\alpha,\gamma}i)i], \theta_{\alpha,\gamma}i)$$
$$= (p^{-1}_{\theta_{\alpha,\gamma}i}\omega^{a}_{\alpha,\gamma}(\theta_{\alpha,\gamma}i), \theta_{\alpha,\gamma}i) = \rho_{\alpha,\gamma}(a)$$

by formulas (5), (6), and (iii). Thus, $\rho_{\beta,\gamma}\rho_{\alpha,\beta} = \rho_{\alpha,\gamma}$.

(C) For arbitrary $\alpha, \beta \in Y$, let $a = (s, i) \in S_\alpha$, $b = (t, j) \in S_\beta$. In view of (5), (6), (i), and (1), we have

$$\rho_{\alpha,\alpha\beta}(a) = (p^{-1}_{(\theta_{\alpha,\alpha\beta}i)(\theta_{\beta,\alpha\beta}j)}\omega^{a}_{\alpha,\alpha\beta}(\theta_{\beta,\alpha\beta}j), \theta_{\alpha,\alpha\beta}i)$$
$$= (p^{-1}_{ij}\,\omega^{a}_{\alpha,\alpha\beta}(ij), \theta_{\alpha,\alpha\beta}i)$$

and $\rho_{\beta,\alpha\beta}(a) = (p^{-1}_{ji}\omega^{b}_{\beta,\alpha\beta}(ji), \theta_{\beta,\alpha\beta}j)$, so

$$\rho_{\alpha,\alpha\beta}(a) * \rho_{\beta,\alpha\beta}(b)$$
$$= (p^{-1}_{ij}\omega^{a}_{\alpha,\alpha\beta}(ij)p_{(\theta_{\beta,\alpha\beta}j)(\theta_{\alpha,\alpha\beta}i)}p^{-1}_{ji}\omega^{b}_{\beta,\alpha\beta}(ji), (\theta_{\alpha,\alpha\beta}i)(\theta_{\beta,\alpha\beta}j))$$
$$= (p^{-1}_{ij}\omega^{a}_{\alpha,\alpha\beta}(ij)\omega^{b}_{\beta,\alpha\beta}(ji), ij) = a * b.$$

Therefore, $S = [Y; S_\alpha, \rho_{\alpha,\beta}]$ which is a Clifford semilattice Y of a family of Rees matrix semigroups $\{\mathcal{U}(M_\alpha, E_\alpha; P_\alpha)\}_{\alpha\in Y}$. ∎

References

1. A.H. Clifford and G.H. Presten, *The Algebraic Theory of Semigroups*, Vol. I, American Mathematical Society, Providence, 1961.
2. B.M. Schein, Bands of semigroups: Variations on a Clifford theme, in: *Semigroup Theory and Its Applications, Proceedings of the 1994 Conference Commemorating the Work of Alfred H. Clifford*, Cambridge University Press, 1996, pp. 53–80.
3. G. Lallement and M. Petrich, A generalization of the Rees theorem in semigroups, *Acta Sci. Math. Szeged* **30** (1969) 113–132.
4. J.M. Howie, *An Introduction to Semigroup Theory*, Academic Press, London, New York, 1976.
5. M. Petrich, Regular semigroups satisfying certain conditions on idempotents and ideals, *Trans. Amer. Math. Soc.* **170** (1972) 245–267.
6. M. Petrich, The structure of completely regular semigroups, *Trans. Amer. Math. Soc.* **189** (1974) 211–236.
7. M. Petrich, *Lectures in Semigroups*, John Wiley & Sons, New York, 1977.