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# Exact Penalty in D.C. Programming

**Le Thi Hoai An[1], Pham Dinh Tao[1], and Le Dung Muu[2]**

[1] *Mathematical Modelling and Applied Optimization Group, Laboratory of Mathematics-INSA Rouen - CNRS URA 1378, BP 08, 76131 Mont Saint Aignan, France*

[2] *Institute of Mathematics, Box 631, Bo Ho, Hanoi, Vietnam*

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**Abstract.** Concave minimization over a bounded polyhedral convex set with an additional reverse convex constraint contains important problems in non-convex programming. Both theoretical and practical studies of this class of non-convex programs can be made more convenient and easier when the reverse convex constraint is penalized. We have proved that if the concave function defining the reverse convex constraint is non-negative over bounded polyhedral convex set, then the exact penalty and the stability of the Lagrangian duality hold. Consequently, equivalent difference of convex ((d.c.) functions) programs are formulated.

## 1. Introduction

Let $K$ be a non-empty bounded polyhedral convex set in $\mathbb{R}^n$ and let $f$, $g$ be concave functions on $K$. We will be concerned with non-convex programs of the following type:

$$\alpha = \inf\{f(x) : x \in K,\ g(x) \le 0\}, \tag{P}$$

where the non-convexity appears both in the objective function and the constraint $\{x \in \mathbb{R}^n : g(x) \le 0\}$.

As in convex optimization, to make the study of (P) easier, we generally try to penalize difficult constraints, that is, $\{x \in \mathbb{R}^n : g(x) \le 0\}$. For example, the usual exact penalty process leads to

$$\alpha_t = \inf\{f(x) + tg^+(x) : x \in K\}, \tag{$\mathrm{P}_t^+$}$$

where $t$ is a positive number (called penalty parameter) and $g^+$ is defined by

$$g^+(x) = \max(0, g(x)), \quad \forall x \in K.$$

Actually, the exact penalty holds if there is a positive number $t_0$ such that, for $t \ge t_0$, the solution sets of (P) and ($\mathrm{P}_t^+$) are identical. Such a property generally holds in convex optimization. However, for non-convex optimization, it becomes more complex when global solutions are involved, and therefore, a specific study should be carried out.

The main contribution of this paper is to state the exact penalty for (P) in case the function $g$ is non-negative over $K$. It is worth noting that under such an assumption, the exact penalty approach and the ordinary Lagrangian duality are identical. It follows that there is the stability of the Lagrangian duality (i.e., there is no duality gap and the solution set of (P) can be deduced from the solution of its dual program (D) as in convex optimization) and the set of exact penalty parameters is nothing but the solution set of (D). This result allows us to transfer (P) into the more suitable d.c. optimization framework [1–3, 14–16, 25–27].

The organization of the paper is as follows. The next section presents a list of important non-convex problems which can be formulated as (P). They consist of convex maximization over the Pareto set, bi-level linear program, linear program with linear complementarity constraint, and mixed zero-one concave minimization programming. Section 3 deals with the proof of the above mentioned result concerning the exact penalty for (P) and the stability of the usual Lagrangian duality relative to this problem. Finally, an equivalent d.c. program of (P) is formulated.

## 2. Non-Convex Programs of the Form (P)

We work, throughout the paper, with the $n$-dimensional Euclidean space $\mathbb{R}^n$ equipped with the canonical inner-product $\langle \cdot, \cdot \rangle$. The set of all proper lower-semicontinuous convex functions on $\mathbb{R}^n$ will be denoted by $\Gamma_0(\mathbb{R}^n)$. For $\varphi \in \Gamma_0(\mathbb{R}^n)$, the effective domain of $\varphi$ is

$$\operatorname{dom} \varphi = \{x \in \mathbb{R}^n : \psi(x) < +\infty\}.$$

If $\psi$ is concave proper and upper-semicontinuous, i.e., $-\psi \in \Gamma_0(\mathbb{R}^n)$, then we set

$$\operatorname{dom} \psi = \operatorname{dom}(-\psi) = \{x \in \mathbb{R}^n : \varphi(x) > -\infty\}.$$

A d.c. program is that of the form

$$\inf\{\theta(x) = \varphi(x) - \psi(x) : x \in \mathbb{R}^n\}$$

with $\varphi, \psi \in \Gamma_0(\mathbb{R}^n)$. Such a function $\theta$ is called d.c. function. D.c. optimization has been extensively developed in recent years from both combinatorial [6, 7, 18, 25–27] nd convex approaches [1–3, 14–16]. It plays a key role in non-convex optimization and almost realistic non-convex programs.

aximization Over the Pareto Set

bounded polyhedral convex set and let $B$ be a $p \times n$ real matrix. ultiple linear objective programming problem

$$\max\{Bx : x \in C\}. \qquad \text{(MLP)}$$

be a *Pareto point* of (MLP) when $x^0$ is in $K$ and $\in K$, then $Bx = Bx^0$. The set of all Pareto points of of (MLP) and is denoted by $C_P$.

Now, let $\varphi$ be a real-valued convex function on $C$ and let us introduce the following non-convex program:

$$\max\{\varphi(x) : x \in C_P\}. \tag{P$_1$}$$

To formulate ($\mathrm{P}_1$) in the form of (P), we consider the function $p$ defined by [3]

$$p(x) = \max\left\{e^T B(y - x) : By \geq Bx,\ y \in C\right\}.$$

It has been proved in [3] that

- $-p$ is proper convex and lower-semicontinuous on $\mathbb{R}^n$ (i.e., $-p \in \Gamma_0(\mathbb{R}^n)$);
- if, in addition, $C$ is polyhedral convex, then $-p$ is polyhedral convex on $\mathbb{R}^n$;
- $\operatorname{dom} p$ is the projection onto $\mathbb{R}^n$ of $\{(x, y) \in \mathbb{R}^n \times C : By \geq Bx\}$;
- $p(x) \geq 0, \forall x \in C$;
- $x \in C_P$ if and only if $x \in C$ and $p(x) = 0$.

Finally, ($\mathrm{P}_1$) admits the following equivalent form of (P):

$$\max\left\{\varphi(x) : p(x) \leq 0,\ x \in C\right\}. \tag{$\tilde{\mathrm{P}}_1$}$$

2.2. Bi-Level Linear Program

Let $K$ and $L$ be non-empty bounded polyhedral convex sets in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Let $a, c \in \mathbb{R}^m$ and $b, d \in \mathbb{R}^n$ be given vectors and consider the following non-convex program:

$$\begin{cases} \max a^T x + b^T y & (\mathrm{P}_2) \\ \text{subject to } x \in K \text{ and } y \text{ solves the linear program} & \\ \max\{c^T x + d^T y : y \in L,\ Ax + By \leq r\}, & (\mathrm{LP}_2) \end{cases}$$

where $A$ is a $p \times m$ real matrix, $B$ is a $p \times n$ real matrix, and $r \in \mathbb{R}^p$. It is the so-called bi-level linear program which is related to the economy concept due to von Stackelberg (see [10] and references therein).

Let $C = \{z = (x, y) \in K \times L : Ax + By \leq r\}$ and let $p_1$ be the function defined on $\mathbb{R}^m$ by

$$p_1(x) := \max\{d^T y : y \in L,\ Ax + By \leq r\}, \quad \forall x \in \mathbb{R}^m.$$

As in Subsec. 2.1, the function $-p_1$ is polyhedral convex on $\mathbb{R}^m$ and finite on the projection of $C$ on $\mathbb{R}^m$.

Consider now the function $p(x, y)$ defined on $\mathbb{R}^m \times \mathbb{R}^n$ by

$$p(x, y) = p_1(x) - d^T y, \quad \forall (x, y) \in \mathbb{R}^m \times \mathbb{R}^n.$$

It is clear that $-p$ is polyhedral convex on $\mathbb{R}^m \times \mathbb{R}^n$, finite and non-negative on $C$. Moreover, $x \in K$ and $y$ solves the linear program ($\mathrm{LP}_2$) if and only if $(x, y) \in C$ and $p(x, y) = 0$. Consequently, the bi-level linear program ($\mathrm{P}_2$) takes the following equivalent form of (P):

$$\max\left\{a^T x + b^T y : (x, y) \in C,\ p(x, y) \leq 0\right\}. \tag{$\tilde{\mathrm{P}}_2$}$$

Remark that if we replace in ($P_2$) the linear function $a^T x + b^T y$ by a convex function $f(x, y)$, the corresponding program ($\tilde{P}_2$) is still of the form (P).

### 2.3. Linear Program with Mixed Linear Complementarity Constraint

Let $A$ be an $n \times n$ real matrix and let $b$, $c \in \mathbb{R}^n$ be given vectors. Let $I$ be a given subset of $\{1, \ldots, n\}$. The problem is defined by

$$\min \left\{ f(x) : Ax \leq b,\ x \geq 0,\ \sum_{i \in I} x_i (b - Ax)_i = 0 \right\}, \tag{$P_3$}$$

where $f$ is a finite concave function on the polyhedral convex set $C := \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$ that is assumed to be bounded. Let $F(x) := b - Ax$ and define $p(x) = \sum_{i \in I} \min\{F_i(x), x_i\}$. Obviously, $-p$ is finite polyhedral convex on $\mathbb{R}^n$ and $p$ is non-negative on $C$ for every $x \in C$. Therefore, ($P_3$) is equivalent to

$$\max \{ f(x) : x \in C,\ p(x) \leq 0 \}. \tag{$\tilde{P}_3$}$$

### 2.4. Mixed Zero-One Concave Minimization Programming

Let $K$ be a non-empty bounded polyhedral convex set in $\mathbb{R}^n$ and let $I$ be a given subset of $\{1, \ldots, n\}$. The mixed zero-one concave minimization programming problem is defined as

$$\min \{ f(x) : x \in K,\ x_i \in \{0, 1\},\ \forall i \in I \}, \tag{$P_4$}$$

where $f$ is a finite concave function on $K$.

Let $C := \{x \in K : 0 \leq x_i \leq 1,\ \forall i \in I\}$ and define

$$p(x) = \sum_{i \in I} x_i (1 - x_i).$$

Clearly, $p$ is a concave quadratic form on $\mathbb{R}^n$ and $p(x) \geq 0$ for every $x \in C$. On the other hand, we have, for $x \in C$,

$$p(x) = 0 \quad \text{if and only if } x_i \in \{0, 1\},\ \forall i \in I.$$

It follows that ($P_4$) can be reformulated in an equivalent way as

$$\min \{ f(x) : x \in C,\ p(x) \leq 0 \}. \tag{$\tilde{P}_4$}$$

## 3. Exact Penalty and Stability of the Lagrangian Duality in Problem (P)

Let us first recall the formulation of (P):

$$\alpha = \inf \{ f(x) : x \in K,\ g(x) \leq 0 \}, \tag{P}$$

in which $K$ is a non-empty bounded polyhedral convex set in $\mathbb{R}^n$, and $f$, $g$ are finite concave functions over $K$. In non-convex programming, it is generally easier to treat the non-convexity in the objective function rather than in the constraints. The exact penalty technique aims to transform (P) into a more tractable equivalent problem of the d.c. optimization framework which has been extensively developed in recent years [1–3, 6, 7, 14–16, 25–27].

For $t \geq 0$, consider the penalized problems [8]:

$$\alpha_+(t) = \inf\left\{f(x) + tg^+(x) : x \in K\right\} \tag{$P_t^+$}$$

with $g^+(x) = \max\{0, g(x)\}$ for every $x \in K$.

Since $g^+$ is a d.c. function on $K$, ($P_t^+$) is a d.c. program. On the other hand, the Lagrangian duality relative to (P) introduces the following d.c. programs [8]:

$$\alpha(t) = \inf\{f(x) + tg(x) : x \in K\}. \tag{$P_t$}$$

The corresponding dual problem is given by

$$\beta = \sup\left\{\alpha(t) : t \in \mathbb{R}^+\right\} \tag{D}$$

with $\mathbb{R}^+$ being the set of non-negative real numbers. It is clear that $\alpha(t) \leq \alpha_+(t)$ for every $t \in \mathbb{R}^+$. If $g$ is non-negative over $K$, then ($P_t^+$) and ($P_t$) are identical: The exact penalty approach for (P) is then nothing but the Lagrangian duality relative to this problem.

In the sequel, the solution sets of (P), ($P_t^+$), and ($P_t$) will be denoted by $\mathcal{P}$, $\mathcal{P}_t^+$, and $\mathcal{P}_t$, respectively.

In this section, we shall discuss relationships between these problems and establish the exact penalty and the stability of the Lagrangian duality in case the function $g$ is non-negative on $K$. Let us first state a general result about the exact penalty in which $K$ is an arbitrary set in $\mathbb{R}^n$ and $f$, $g$ are arbitrary real-valued functions defined on $K$.

Clearly, $\alpha_+(0) \leq \alpha$. There are two cases to be distinguished:

(a) the constraint $g(x) \leq 0$ is essential in (P): $\alpha_+(0) < \alpha$. In this case we have

$$\emptyset \neq \{x \in K : f(x) < \alpha\} \subset \{x \in K : g(x) > 0\}. \tag{1}$$

(b) $\alpha_+(0) = \alpha$. This case occurs if and only if either $\{x \in K : g(x) > 0\}$ is empty (i.e., $K = \{x \in K : g(x) \leq 0\}$) or $\inf\{f(x) : x \in K,\ g(x) > 0\} \geq \alpha$. In other words, under the usual convention $\inf \emptyset = +\infty$,

$$\alpha_+(0) = \alpha \Leftrightarrow \inf\{f(x) : x \in K,\ g(x) > 0\} \geq \alpha. \tag{2}$$

**Theorem 1.**

(i) *$\alpha_+$ is an increasing function of $t \in \mathbb{R}^+$ and is bounded above by $\alpha$.*

(ii) *If $\alpha_+(0) < \alpha$, then $\alpha_+(t) = \alpha$ if and only if*

$$t_0 = \sup\left\{\frac{\alpha - f(x)}{g(x)} : x \in K,\ f(x) < \alpha\right\} \text{ is finite and } t \geq t_0.$$

(iii) *If $\alpha_+(0) = \alpha$, then the corresponding $t_0$ is zero and $\alpha^+(t) = \alpha$ for every $t \geq t_0$.*

*If $t_0$ is finite and $\mathcal{P}$ is non-empty, then*

(iv) $\mathcal{P} \cap \mathcal{P}_t^+ \neq \emptyset \Leftrightarrow \mathcal{P} \subset \mathcal{P}_t^+ \Leftrightarrow \alpha_+(t) = \alpha$.

(v) $\mathcal{P}_t^+ = \mathcal{P}$ *if $t > t_0$.*

*Proof.* We have for $t \in \mathbb{R}^+$

$$\alpha_+(t) = \inf\left[\inf\{f(x) + tg^+(x) : x \in K, g(x) \le 0\}, \inf\{f(x) + tg^+(x) : x \in K, g(x) > 0\}\right]$$
$$= \inf\left[\inf\{f(x) : x \in K, g(x) \le 0\}, \inf\{f(x) + tg^+(x) : x \in K, g(x) > 0\}\right]. \quad (3)$$

So $\alpha_+(t) \le \alpha$. The rest of (i) is straightforward.

(ii) It follows from (3) that $\alpha_+(t) = \alpha$ if and only if

$$\inf\{f(x) + tg(x) : x \in K, g(x) > 0\} \ge \alpha,$$

i.e.,

$$tg(x) \ge \alpha - f(x), \ \forall x \in K, \ g(x) > 0. \quad (4)$$

Since $\alpha_+(0) < \alpha$, according to (1), the relation (4) is equivalent to

$$t \ge t_0 = \sup\left\{\frac{\alpha - f(x)}{g(x)} : x \in K, \ g(x) > 0\right\}.$$

But $t_0 \ge 0$, then we can write

$$t_0 = \sup\left\{\frac{\alpha - f(x)}{g(x)} : x \in K, \ f(x) < \alpha\right\}$$

and (ii) is proved.

The proof of (iii) follows by using (2) and the same reasonings as above.

Now, let $x_t^+ \in \mathcal{P} \cap \mathcal{P}_t^+$. Then

$$\alpha_+(t) = f(x_t^+) + tg^+(x_t^+) = f(x_t^+) \ge \alpha.$$

But $\alpha_+(t) \le \alpha$ by (i), hence, $\alpha_+(t) = \alpha$. On the other hand, we have for $x \in \mathcal{P}$

$$f(x) + tg^+(x) = f(x) = \alpha = \alpha_+(t),$$

i.e., $x \in \mathcal{P}_t^+$.

The converse part is straightforward since $\mathcal{P} \subset \mathcal{P}_t^+$ when $\alpha_+(t) = \alpha$. So it remains to prove that $\mathcal{P}_t^+ = \mathcal{P}$ if $t > t_0$. Let $t > t_0$. Then according to the just shown result, we have $\mathcal{P} \subset \mathcal{P}_{t_0}^+$ and $\mathcal{P} \subset \mathcal{P}_t^+$. Hence, proving $\mathcal{P} = \mathcal{P}_t^+$ amounts to proving $g(x) \le 0$ for every $x \in \mathcal{P}_t^+$. If it is not the case, then there is $x \in \mathcal{P}_t^+$ such that $g(x) > 0$. It implies

$$\alpha_+(t) = f(x) + tg^+(x) = f(x) + tg(x) > f(x) + t_0 g(x) \ge \alpha_+(t_0) = \alpha.$$

It is in contradiction to (i). ■

Now, let us briefly discuss the stability of the Lagrangian duality relative to (P).

**Proposition 1.**

(i) *$\alpha(t) \le \alpha_+(t) \le \alpha$ for every $t \in \mathbb{R}^+$; $\alpha_+(0) = \alpha(0)$; $\mathcal{P}_0^+ = \mathcal{P}_0$. For $t > 0$, if $\alpha_+(t) = \alpha(t)$, then $\mathcal{P}_t^+ = \{x \in \mathcal{P}_t : g(x) \ge 0\}$.*

(ii) *For $t > 0$, if $\alpha(t) = \alpha$, then $\alpha_+(t) = \alpha(t) = \alpha$. This possibility occurs only if $t_0$ (in Theorem 1) is finite and $t \ge t_0$. In this case, we have $t \in \mathcal{D}$, $\mathcal{P}_t^+ = \{x \in \mathcal{P} : g(x) \ge 0\}$. In particular, if such a $t$ is greater than $t_0$, $(t > t_0)$, then*

$$\mathcal{P}_t^+ = \mathcal{P} = \{x \in \mathcal{P}_t : g(x) = 0\}.$$

*Proof.* (i) Let $t > 0$ verify $\alpha_+(t) = \alpha(t)$. Since

$$f(x) + tg^+(x) \geq f(x) + tg(x) \geq \alpha(t) \quad \forall x \in K,$$

$x \in \mathcal{P}_t^+$ if and only if $x \in \mathcal{P}_t$ and $tg^+(x) = tg(x)$, i.e., $\mathcal{P}_t^+ = \{x \in \mathcal{P}_t : g(x) \geq 0\}$.

(ii) is straightforward from Theorem 1 and the complementarity property for the Lagrangian duality is without gap. ∎

*Remark 1.* (i) If $g(x)$ is non-negative on $K$, then the exact penalty approach and the Lagrangian duality relative to (P) are the same.

(ii) Theorem 1 states that the exact penalty holds if and only if $t_0$ is finite.

We are now in the position to prove the main result.

**Theorem 2.** *Let $K$ be a non-empty, bounded polyhedral convex set in $\mathbb{R}^n$ and let $f$, $g$ be finite concave on $K$. Assume the feasible set of (P) be non-empty and $g$ be non-negative on $K$. Then the problems ($\mathrm{P}_t$) and ($\mathrm{P}_t^+$) are identical. Furthermore,*

(i) *If the vertex set of $K$, $V(K)$, is contained in $\{x \in K : g(x) \leq 0\}$, then $t_0 = 0$.*
(ii) *If $V(K)$ is not contained in $\{x \in K : g(x) \leq 0\}$, then*

$$t_0 \leq \frac{f(x^0) - \alpha(0)}{S} \quad \textit{for every } x^0 \in K,\ g(x^0) \leq 0$$

*where* $S := \min\{g(x) : x \in V(K),\ g(x) > 0\}$.
(iii) *The solution set $\mathcal{D}$ of the dual problem (D) is $[t_0, +\infty[$.*

*Proof.* First, note that the exact penalty approach and the Lagrangian duality relative to (P) are identical since the function $g(x)$ is assumed to be non-negative on $K$.

(i) We have

$$V(K) \subset \{x \in K : g(x) \leq 0\} \subset K.$$

It implies

$$\min\{f(x) : x \in V(K)\} \geq \min\{f(x) : x \in K,\ g(x) \leq 0\} \geq \min\{f(x) : x \in K\}.$$

Since $f$ is concave on $K$,

$$\alpha(0) = \min\{f(x) : x \in K\} = \min\{f(x) : x \in K,\ g(x) \leq 0\} = \alpha.$$

(ii) Consider now the case where $V(K)$ is not contained in $\{x \in K : g(x) \leq 0\}$. For this, $S > 0$. Since $f$ and $g^+ = g$ are finite concave on $K$, we have

$$\alpha(t) = \min\{f(x) + tg(x) : x \in K\} = \min\{f(x) + tg(x) : x \in V(K)\}.$$

Now, let $t > (f(x^0) - \alpha(0))/S$, where $x^0 \in K$, $g(x^0) \leq 0$ and let $x_t \in V(K)$ be a solution to ($\mathrm{P}_t$). We have

$$f(x_t) + tg(x_t) \leq f(x) + tg(x), \quad \forall x \in K.$$

In particular,

$$f(x_t) + tg(x_t) \leq f(x^0) + tg(x^0).$$

So,

$$tg(x_t) \leq f(x^0) - f(x_t) \leq f(x^0) - \alpha(0). \tag{5}$$

It follows that $g(x_t) \leq 0$. In fact, if $g(x_t) > 0$, then $g(x_t) \geq S$ and (5) implies

$$t \leq \frac{f(x^0) - \alpha(0)}{g(x_t)} \leq \frac{f(x^0) - \alpha(0)}{S},$$

a contradiction. Since $g(x_t) \leq 0$, we have $x_t \in \mathcal{P}$ and $t_0 \leq (f(x^0) - \alpha(0))/S$ by virtue of Theorem 1.

(iii) is straightforward. ■

*Remark 2.* (i) It is worth noting that if $g$ is concave on $K$ and if the feasible set of (P) is non-empty, then $V(K) \cap \{x \in K : g(x) \leq 0\} \neq \emptyset$. Indeed, if $V(K) \subset \{x \in K : g(x) > 0\}$, then $K \subset \{x \in K : g(x) > 0\}$ because $g$ is concave on $K$. It thus implies the emptiness of the feasible set of (P).

(ii) If $f$ is concave on $K$ and $V(K) \subset \{x \in K : g(x) \leq 0\}$, then $\alpha(0) = \alpha$. In this case the constraint $g(x) \leq 0$ is not essential for (P).

(iii) If the functions $f$ and $g$ are finite concave on a non-empty, bounded polyhedral convex set $K$ and if $g$ is non-negative on $K$, then ($\mathrm{P}_t$) and ($\mathrm{P}_t^+$) are identical and belong to the d.c. optimization framework

$$\begin{aligned}\alpha(t) &= \min\{f(x) + tg(x) : x \in K\} \qquad (\mathrm{P}_t)\\ &= \min\left\{\chi_K(x) - [-f(x) - tg(x)] : x \in \mathbb{R}^n\right\}\\ &= \min\left\{\varphi(x) - \psi(x) : x \in \mathbb{R}^n\right\}\end{aligned}$$

Here, $\psi = -(f+tg)$ and $\varphi = \chi_K$ are convex functions on $\mathbb{R}^n$, $\chi_K$ stands for the indicator function of $K$: $\chi_K(x) = 0$ if $x \in K$, and $+\infty$ otherwise. According to Theorem 2, there is a finite, non-negative number $t_0$ such that, for $t > t_0$, $(P_t)$ is a d.c. program equivalent to (P).

Let us close the paper by the following result which is useful to computational methods.

**Proposition 2.** *Under the assumptions of Theorem 2, we have*

$$\begin{aligned}\alpha &= \min\{f(x) : x \in K,\ g(x) \leq 0\}\\ &= \min\{f(x) : x \in V(K),\ g(x) \leq 0\}.\end{aligned}$$

*Proof.* We have for $t \geq 0$

$$\alpha(t) = \min\{f(x) + tg(x) : x \in K\}.$$

Since $f$ and $g$ are concave on $K$,

$$\begin{aligned}\alpha(t) &= \min\{f(x) + tg(x) : x \in V(K)\}\\ &= \min\big[\min\{f(x) + tg(x) : x \in V(K), g(x) \leq 0\},\\ &\qquad \min\{f(x) + tg(x) : x \in V(K), g(x) > 0\}\big]. \end{aligned} \tag{6}$$

If $\{x \in V(K) : g(x) > 0\} = \emptyset$ (i.e., $V(K) \subset \{x \in K,\ g(x) \le 0\}$), then

$$\begin{aligned}\alpha(t) &= \min\{f(x) : x \in V(K),\ g(x) \le 0\}\\ &\ge \min\{f(x) : x \in K,\ g(x) \le 0\} = \alpha.\end{aligned}$$

But $\alpha(t) \le \alpha$ by virtue of Theorem 1, so

$$\min\{f(x) : x \in K,\ g(x) \le 0\} = \min\{f(x) : x \in V(K),\ g(x) \le 0\}.$$

If $\{x \in V(K) : g(x) > 0\} \ne \emptyset$, then $\alpha(t) \ge \alpha$ when

$$\min\{f(x) + tg(x) : x \in V(K),\ g(x) > 0\} > \alpha. \tag{7}$$

Indeed, we then have, according to (6), $\alpha(t) \ge \alpha$ since

$$\begin{aligned}\min\{f(x) + tg(x) : x \in V(K), g(x) \le 0\} &= \min\{f(x) : x \in V(K),\ g(x) \le 0\}\\ &\ge \min\{f(x) : x \in K, g(x) \le 0\} = \alpha.\end{aligned}$$

It then follows from Theorem 1 that $\alpha(t) = \alpha$.

Now, translate condition (7) into

$$f(x) + tg(x) > \alpha, \quad \forall x \in V(K),\ g(x) > 0,$$

i.e.,

$$t > \max\left\{\frac{\alpha - f(x)}{g(x)} : x \in V(K),\ g(x) > 0\right\}.$$

Finally, with such a $t$, we obtain

$$\alpha(t) = \alpha = \min\{f(x) : x \in V(K),\ g(x) \le 0\}.$$

■

*Remark 3.* Under the assumptions of Theorem 2, it is easy to show that

$$\{x \in K : g(x) \le 0\} \subset co\{x \in V(K) : g(x) \le 0\}, \tag{8}$$

where *co* stands for the convex hull. Moreover, if $g$ is strictly concave on $K$, then $\{x \in K : g(x) \le 0\} = \{x \in V(K) : g(x) \le 0\}$. Proposition 2 can then also be deduced from (8).

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