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Short Communication

Some Geometrical Properties of the Spectrum of Functions of Orlicz Type*

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The study of properties of functions in connection with the support of their Fourier transform has been considered by Schwartz, Hörmander, Bang, and many other mathematicians (see [1,4,7] and references there). In particular, the spectrum of functions in Orlicz spaces has been studied by Bang [2].

The spectrum of f is, by definition, the support of its Fourier transform $\hat{f}$ (see [5,7]). Denote $\mathrm{sp}(f) = \mathrm{supp}\,\hat{f}$. Then the geometry of $\mathrm{sp}(f)$, in general, can have arbitrary character. In this paper, by modifying the method of [2], we give some geometrical properties of the spectrum of functions in $N_\Phi(\mathbb{R}^n)$, Φ being a concave function (see [6,8]). Note that the Orlicz spaces are generated by convex functions. Here, we have to overcome some technical difficulties due to the difference between convex and non-convex functions.

Let C denote the family of all non-zero concave functions $\Phi : [0, +\infty) \to [0, +\infty]$, which are non-decreasing and satisfy $\Phi(0) = 0$. For an arbitrary measurable function f and $\Phi \in C$, we define

$$\|f\|_{N_\Phi} = \int_0^\infty \Phi\big(\lambda_f(t)\big)dt,$$

where $\lambda_f(t) = \mu(\{x : |f(x)| > t\})$, $t \geq 0$. Let $N_\Phi = N_\Phi(\mathbb{R}^n)$ be the space of all measurable functions f such that $\|f\|_{N_\Phi} < \infty$. Then N_Φ is a Banach space [8].

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Denote by $M_\Phi = M_\Phi(\mathbb{R}^n)$ the space of measurable functions g such that

$$\|g\|_{M_\Phi} = \sup\left\{\frac{1}{\Phi(\mu(E))}\int_E |g(x)|dx : \ E \subset \mathbb{R}^n, \ 0 < \mu(E) < \infty\right\} < \infty.$$

Then M_Φ is a Banach space too [8].

We need the following results:

Lemma 1. [3] *If $f \in N_\Phi$ and $h \in L_1(\mathbb{R}^n)$, then $f * h \in N_\Phi$ and $\|f * h\|_{N_\Phi} \le \|f\|_{N_\Phi}\|h\|_1$.*

Lemma 2. *Let $f \in N_\Phi(\mathbb{R}^n)$. If $\mathrm{sp}(f)$ is bounded, then f is bounded.*

Proof. Since the spectrum of f is bounded, we can choose $\hat{\psi}(\xi) \in C_0^\infty(\mathbb{R}^n)$, such that $\hat{\psi} = 1$ in some neighborhood of $\mathrm{sp}(f)$, to obtain

$$\begin{aligned}\|(F^{-1}\hat{f})(x)\|_\infty &= \|(F^{-1}(\hat{\psi}\hat{f}))(x)\|_\infty \\ &= \|\psi * f\|_\infty \le \|\psi\|_1\|f\|_{N_\Phi} < \infty. \quad \blacksquare\end{aligned}$$

By applying Lemmas 1 and 2 and using the techniques of the proof of Theorems 1 and 2 in [2] for spaces N_Φ, we obtain the following results:

Theorem 1. *Let $\Phi(t) \in \mathcal{C}$, $f \in N_\Phi$, $f(x) \not\equiv 0$ and let $\xi^0 \in \mathrm{sp}(f)$ be an arbitrary point. Then the restriction of $\hat{f}$ to any neighborhood of ξ^0 cannot concentrate on any finite number of hyperplanes.*

The proof of the theorem is based on Theorem 2.3.5 in [4] for the distributions with compact support and the properties of the Fourier transform.

Corollary 1. *Let $\Phi \in \mathcal{C}$, $f \in N_\Phi$, and $f(x) \not\equiv 0$. Then for any $\xi^0 \in \mathrm{sp}(f)$, there exists a sequence $\{\xi^m\} \subset \mathrm{sp}(f)$ such that $\xi_j^m \neq \xi_j^0$, $j = 1, \dots, n$ and $\xi^m \to \xi^0$.*

Corollary 2. *Let $\Phi \in \mathcal{C}$, $f \in N_\Phi$, and $f(x) \not\equiv 0$. Then for any $\xi^0 \in \mathrm{sp}(f)$, there exists a sequence $\{\xi^m\} \subset \mathrm{sp}(f)$ such that $\xi_j^m \neq 0$, $j = 1, \dots, n$ and $\xi^m \to \xi^0$.*

Corollary 3. *Let $\Phi \in \mathcal{C}$, $f \in N_\Phi$, and $f(x) \not\equiv 0$. Then*

$$\mathrm{span}(\mathrm{sp}(f)) = \mathrm{span}(\mathrm{sp}(f) \setminus \{\xi^0\}) = \mathbb{R}^n$$

for any $\xi^0 \in \mathrm{sp}(f)$.

Corollary 4. *One has $\mathrm{sp}(D^\alpha f) \subset \mathrm{sp}(f)$, where $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = -i\partial/\partial x_j$. Further, if the hypotheses of Corollary 1 are satisfied, then $\mathrm{sp}(D^\alpha f) = \mathrm{sp}(f)$.*

Theorem 2. *Let $\Phi \in \mathcal{C}$, $f \in N_\Phi$ and let $\alpha \ge 0$ be a multi-index. In order that $\sup_{\mathrm{sp}(f)} |\xi^\alpha| = 0$, it is necessary and sufficient that $D^\alpha f(x) \equiv 0$, where $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = -i\partial/\partial x_j$.*

Sketch of proof. We first prove that $\xi^\alpha \hat{f}(\xi)$ concentrates on the plane $\xi_1 = \cdots = \xi_k = 0$.

Next, we show that, if $\xi^\alpha \psi(\xi)\hat{f}(\xi)$ concentrates on the plane $\xi_1 = \cdots = \xi_k = 0$, then $D^\alpha F^{-1}\psi * f(x) \equiv 0$.

Finally, for every $\varphi \in C_0^\infty(\mathbf{R}^n)$, we choose $\psi \in C_0^\infty(\mathbf{R}^n)$ such that $\psi = 1$ in some neighborhood of suppφ. Then

$$\begin{aligned}\langle D^\alpha f, \hat{\varphi}\rangle &= \langle \xi^\alpha \hat{f}(\xi), \varphi(\xi)\rangle = \langle \xi^\alpha \psi(\xi)\hat{f}(\xi), \varphi(\xi)\rangle \\ &= \langle D^\alpha F^{-1}\psi * f, \hat{\varphi}\rangle = \langle 0, \hat{\varphi}\rangle = 0.\end{aligned}$$

So it follows from the density of $C_0^\infty(\mathbf{R}^n)$ in $\mathcal{S}$ that $\langle D^\alpha f, \hat{\varphi}\rangle = 0$ for all $\varphi \in \mathcal{S}$. Therefore, $D^\alpha f(x) \equiv 0$. ∎

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References

1. Ha Huy Bang, Functions with bounded spectrum, *Trans. Amer. Math. Soc.* **347** (1995) 1067–1080.
2. Ha Huy Bang, Spectrum of functions in Orlicz spaces, *J. Math. Sci. Univ. Tokyo* **4** (1997) 341–349.
3. H. H. Bang and H. M. Le, On the Kolmogorov–Stein inequality, *J. Ineq. Appl.* **4** (1998) 1–8.
4. L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer–Verlag, Berlin–Heidelberg, 1983.
5. L. Hörmander, A new proof and a generalization of an inequality of Bohr, *Math. Scand.* **2** (1954) 33–45.
6. M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker Inc., New York, 1995.
7. L. Schwartz, *Théorie des Distributions II*, Hermann, Paris, 1951.
8. M. S. Steigerwalt and A. J. White, Some function spaces related to L_p, *Proc. London. Math. Soc.* **22** (1971) 137–163.