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Asymptotic and Oscillatory Behavior of Higher-Order Nonlinear Neutral Difference Equation

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Abstract. This paper studies oscillation and asymptotic behavior of higher-order nonlinear forced neutral difference equations. We obtain a series of sufficient conditions for the oscillation and the asymptotic behavior of solutions of higher-order neutral difference equations.

1. Introduction

Recently, there has been some activity concerning the study of the oscillatory and asymptotic behavior of the solutions of higher-order neutral delay difference equations (see, for example, [1–10] and the references cited therein).

In particular, Graef et al. [3] studied the following difference equation:

$$\Delta^m [y_{n-m+1} + p_{n-m+1} y_{n-m+1-k}] + \delta F(n, y_{n-1}) = 0.$$

Zafer [10] studied a more general difference equation of the form

$$\Delta\big[a(t)\Delta^{n-1}(x(t) + p(t)x(\tau(t)))\big] + F(t, x(\sigma(t))) = 0, \quad t \in I,$$

where I is the discrete set $\{0, 1, 2, \dots\}$ and Δ is the forward difference operator $\Delta x(t) = x(t+1) - x(t)$.

In this paper, we are concerned with a more general nonlinear forced difference equation of the form

$$\Delta^m (x_n - p_n x_{n-\tau(n)}) - \sum_{i=1}^{s} Q_i(n) f_i(x_{n-\sigma_i(n)}) = h_n \,. \tag{1}$$

Here, $m \geq 2$ is even, $\{p_n\}$ is a positive real sequence, $\{Q_i(n)\}$ is a non-negative real sequence, $\{\tau(n)\}$ is a given positive sequence of integer with $\lim_{n\to\infty}(n-\tau(n)) = \infty$, $\{\sigma_i(n)\}$ are non-negative sequences of integer with $\lim_{n\to\infty}(n-\tau(n)) = \infty$ for $i = 1, 2, \ldots, s$. $\{h_n\}$ is a real sequence which is oscillatory. Moreover, there is at least an integer j, $1 \leq j \leq k$, such that $\sigma_j(n) > 0$ and $\tau(n) > 0$, $f_i(u) \in C(R, R)$ and non-decreasing, $uf_i(u) > 0$ for $u \neq 0$ and $i = 1, 2, \ldots, s$.

By a solution of (1), we mean a real sequence $\{x_n\}$ which satisfies Eq. (1) for $n \geq 0$. A solution $\{x_n\}$ of (1) is said to be eventually positive if $x_n > 0$ for all large n, and eventually negative if $x_n < 0$ for all large n. It is said to be oscillatory if it is neither eventually positive nor eventually negative. We will also say that (1) is oscillatory if each of its solution is oscillatory.

Throughout this paper, we assume that there exists an oscillatory sequence $\{r_n\}$ such that $\Delta^m r_n = h_n$ for $n = 1, 2, \ldots$, and $\lim_{n\to\infty} r_n = 0$. Set

$$z_n = x_n - p_n x_{n-\tau(n)} - r_n . \tag{2}$$

By convention, empty sums will be taken to be zero.

In this paper, we will give some sufficient conditions of oscillatory and asymptotic behavior of solutions of Eq. (1) by using a method which differs from [3,9].

2. Main Results

Lemma 1. [1] *Let $\{y_n\}$ be a sequence of positive real numbers in $N = \{0, 1, 2, \ldots\}$, and $\Delta^m y_n \leq 0$. Let $\Delta^m y_n$ be of constant sign with $\Delta^m y_n$ not being identically zero on any subset $\{n_0, n_0+1, \ldots\}$ of N. Then there exists an integer l, $0 \leq l \leq m-1$, with $m+l$ odd for $\Delta^m y_n \leq 0$, and $m+l$ even for $\Delta^m y_n \geq 0$ such that*

$$l \leq m-1 \text{ implies } (-1)^{l+k}\Delta^k y_n > 0, \quad \text{for all } n \geq N,\ l \leq k \leq m-1,$$

and

$$l \geq 1 \text{ implies } \Delta^k y_n > 0, \quad \text{for all } n \geq N,\ 1 \leq k \leq l-1.$$

Lemma 2. *Let $0 < p_n \leq B$ for $n \geq n_0$ and some positive constant B. Assume that there is at least an integer j, $1 \leq j \leq s$, such that $\sum_{n=n_0}^{\infty} Q_j(n) = \infty$. If $\{x_n\}$ is a bounded solution of Eq. (1) and $\{x_n\}$ is eventually positive (or negative), then $\lim_{n\to\infty} z_n = 0$. Moreover, for all large n, we have $(-1)^k\Delta^k z_n > 0$ (or < 0) for $k = 1, 2, \ldots, m$.*

Proof. Let $\{x_n\}$ be an eventually positive bounded solution of Eq. (1) (the proof when $\{x_n\}$ is eventually negative is similar), and without loss of generality, we may assume that $x_n > 0$, $x_{n-\tau(n)} > 0$ for $i = 1, 2, \ldots, s$ and $n \geq n_1 \geq n_0$. Since $\{x_n\}$ is bounded and $\lim_{n\to\infty} r_n = 0$, so that, by (2), there is a $n_2 \geq n_1$ such that $\{z_n\}$ is bounded for $n \geq n_2$. By (1) and (2), we have

$$\Delta^m z_n = \sum_{i=1}^{s} Q_i(n) f_i(x_{n-\sigma_i(n)}) > 0 \text{ for } n \geq n_2. \tag{3}$$

It follows that $\Delta^k z_n$ $(k = 0, 1, \ldots, m-1)$ are strictly monotone and are eventually of constant sign. Since $\{z_n\}$ is bounded, we may set $\lim_{n\to\infty} z_n = L$ $(-\infty < L < \infty)$. First, suppose $-\infty < L < 0$. In view of $\lim_{n\to\infty} r_n = 0$, then there exists a constant $c > 0$ and a $n_3 \ge n_2$, such that $z_n < -c < 0$ for $n \ge n_3$. Since $n \ge n_3$, $\Delta^m z_n > 0$, $z_n < 0$ and $\{z_n\}$ is bounded, set $y_n = -z_n$. Then as $n \ge n_3$, $y_n > 0$, $\Delta^m y_n = -\Delta^m z_n < 0$ and $\{y_n\}$ is bounded. Observe that m is even. By Lemma 1, there exist a $n_4 \ge n_3$ and an integer $l \in \{1, 3, 5, \ldots, m-1\}$, such that, as $n \ge n_4$,

$$\Delta^k y_n > 0 \text{ for } k = 0, 1, 2, \ldots, l-1,$$

and

$$(-1)^{k+l}\Delta^k y_n > 0 \text{ for } k = l, l+1, \ldots, m-1. \tag{4}$$

Since $\{y_n\}$ is bounded, we may show that $l = 1$. Otherwise, if $l \ge 3$, then, by (4), we have $y_n > 0$, $\Delta y_n > 0$ and $\Delta^2 y_n > 0$ for $n \ge n_4$. So Δy_n is strictly increasing, hence, there exist a $n_5 \ge n_4$ and a constant $c > 0$, such that $\Delta y_n > c > 0$ for $n \ge n_5$. By summing from n_5 to n and letting $n \to \infty$, we have $\lim_{n\to\infty} y_n = \infty$, which contradicts the fact that $\{y_n\}$ is bounded. Hence, $l \ge 3$ is impossible. So $l = 1$ holds. From (4), we have, as $n \ge n_4$, $y_n > 0$ and $(-1)^{k+l}\Delta^k y_n > 0$ for $k = 1, 2, \ldots, m-1$, that is, as $n \ge n_4$, $z_n < 0$ and $(-1)^k\Delta^k z_n > 0$ for $k = 1, 2, \ldots, m-1$. In particular, $\Delta^{m-1} z_n < 0$ for $n \ge n_4$. Since $\{x_n\}$ is bounded, we set $\lim_{n\to\infty} \inf x_n = a$ $(0 \le a < \infty)$. We wish to show that $a > 0$. Otherwise, if $a = 0$, then there is an integer sequence $\{n_i\}$, such that $\lim_{i\to\infty} n_i = \infty$ and $\lim_{n\to\infty} \inf x_n = a = 0$.

By (2), we have

$$x_{n_i+\tau(n_i)} = z_{n_i+\tau(n_i)} + p_{n_i+\tau(n_i)} x_{n_i} + r_{n_i+\tau(n_i)}.$$

So let $i \to \infty$, we have $\lim_{i\to\infty} \inf x_{n_i+\tau(n_i)} = d < 0$. This contradicts $x_n > 0$ for $n \ge n_1$. Hence, $a > 0$ holds, that is, $\lim_{n\to\infty} x_n = a > 0$. It follows that there are a constant $c_1 > 0$ and a $n_5 \ge n_4$, such that $x_n > c_1 > 0$ and $x_{n-\sigma_i(n)} > c_1 > 0$ for $n \ge n_5$. So, by (3), as $n \ge n_5$, we have

$$\Delta^m z_n \ge \sum_{i=1}^{s} Q_i(n) f_i(c_1) \ge b \sum_{i=1}^{s} Q_i(n) \ge b Q_j(n), \tag{5}$$

where $b = \min_{1\le i\le s}\{f_i(c_1)\} > 0$.

By summing (5) from n_5 to n and letting $n \to \infty$, we have $\lim_{n\to\infty} \Delta^{m-1} z_n = \infty$. This contradicts $\Delta^{m-1} z_n < 0$ for $n \ge n_4$. Hence, the inequality $-\infty < L < 0$ cannot occur.

If $0 < L < \infty$, then there exist a constant $c > 0$ and a $n_3 \ge n_2$, such that $z_n > c > 0$ for $n \ge n_3$. Since $n \ge n_2$, $\Delta^m z_n > 0$, and $\{z_n\}$ is bounded, observe that m is even. By Lemma 1, there exists a $n_4 \ge n_3$ and $l = 0$, such that $(-1)^k\Delta^k z_n > 0$ for $n \ge n_4$ and $k = 0, 1, 2, \ldots, m-1$. In particular, $\Delta^{m-1} z_n < 0$ for $n \ge n_4$. Observe that $z_n > c > 0$ for $n \ge n_3$ and $\lim_{n\to\infty} r_n = 0$. Hence, there exists a constant $c_1 > 0$ and $n_5 \ge n_4$, such that $z_n + r_n > c_1 > 0$ for $n \ge n_5$. By (2), we have $x_n > z_n + r_n > c_1 > 0$ for $n \ge n_5$. So we may take a $n_6 \ge n_5$, such that $x_{n-\sigma_i(n)} > c_1 > 0$ for $n \ge n_6$ and $i = 1, 2, \ldots, s$. From (3), we obtain

$$\Delta^m z_n \ge \sum_{i=1}^{s} Q_i(n) f_i(c_1) \ge b \sum_{i=1}^{s} Q_i(n) \ge b Q_j(n), \ n \ge n_6, \tag{6}$$

where $b = \min_{1\le i\le s}\{f_i(c_1)\} > 0$.

By summing (6) from n_6 to n and letting $n \to \infty$, we have $\lim_{n\to\infty} \Delta^{m-1} z_n = \infty$. This contradicts $\Delta^{m-1} z_n < 0$ for $n \ge n_4$. Hence, $0 < L < \infty$ is impossible. So $L = 0$ holds, that is, $\lim_{n\to\infty} z_n = 0$.

Since $\lim_{n\to\infty} z_n = 0$, it is not difficult to show by contradiction that $\lim_{n\to\infty} \Delta^k z_n = 0$ for $k = 1, 2, \ldots, m-1$. Since $\Delta^m z_n > 0$ for $n \ge n_2$ and m is even, hence, it is easy to see that, for sufficiently large n, $(-1)^k \Delta^k z_n > 0$ for $k = 0, 1, 2, \ldots, m$. The proof is complete. ■

Theorem 1. *Assume that the following conditions are satisfied:*

(c_1) $0 < p_n \le B$ *for* $n \ge n_0$ *and some positive constant* $B, 0 < B < 1$.

(c_2) *There exists at least an integer* j, $1 \le j \le s$, *such that*

$$\sum_{n=n_0}^{\infty} Q_i(n) = \infty.$$

Then every bounded non-oscillatory solution of Eq. (1) tends to zero as $n \to \infty$.

Proof. Let $\{x_n\}$ be a bounded non-oscillatory solution of Eq. (1). Without loss of generality, assume that $\{x_n\}$ is eventually positive (the proof when $\{x_n\}$ is eventually negative is similar). By Lemma 2, we have $\lim_{n\to\infty} z_n = 0$. Since $\lim_{n\to\infty} r_n = 0$, so $\lim_{n\to\infty}(z_n + r_n) = 0$. Observe that $\{x_n\}$ is bounded, hence, we set $\lim_{n\to\infty} \sup x_n = a$, then $0 \le a < \infty$. We wish to show that $a = 0$. Otherwise, if $a > 0$, then there is an integer sequence $\{n_k\}$, such that $\lim_{k\to\infty} n_k = \infty$ and $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} \sup x_n = a > 0$. Since $\{x_n\}$ is bounded, so $\{x_{n_k - \tau(n_k)}\}$ is bounded. Hence, there exists a sequence $\{n_{k_i}\} \subset \{n_k\}$, such that $\lim n_{k_i} = \infty$ and $\lim_{i\to\infty} x_{n_{k_i} - \tau(n_{k_i})}$ exists. By (2), we have

$$z_{n_{k_i}} + r_{n_{k_i}} = x_{n_{k_i}} - p_{n_{k_i}} x_{n_{k_i} - \tau(n_{k_i})} \ge x_{n_{k_i}} - B x_{n_{k_i} - \tau(n_{k_i})},$$

so that

$$\begin{aligned} 0 = \lim_{i\to\infty} (z_{n_{k_i}} + r_{n_{k_i}}) &\ge \lim_{i\to\infty} x_{n_{k_i}} - B \lim_{i\to\infty} x_{n_{k_i} - \tau(n_{k_i})} \\ &= a - B \lim_{i\to\infty} x_{n_{k_i} - \tau(n_{k_i})}. \end{aligned}$$

Hence, we have $\lim_{i\to\infty} x_{n_{k_i} - \tau(n_{k_i})} \ge a/B > a$. This contradicts $\lim_{n\to\infty} \sup x_n = a$. Hence, $a > 0$ is impossible, and so $a = 0$ holds, that is, $\lim_{n\to\infty} \sup x_n = a$ holds. Observe that $x_n > 0$ eventually, so that $\lim_{n\to\infty} x_n = 0$. The proof is complete. ■

Theorem 2. *Let (c_1), (c_2) be satisfied. Moreover, assume that the following conditions hold:*

(c_3) *There exists a positive constant* λ, *such that*

$$\liminf_{u\to 0} \frac{f_i(u)}{u} \ge \lambda \ \textit{for}\ i = 1, 2, \ldots, s,$$

(c_4)

$$\limsup_{n\to\infty} \sum_{w=n_0}^{n} w^{m-1} Q_j(w) r_{w-\sigma_j(w)} = \infty,$$

(c$_5$)

$$\liminf_{n\to\infty} \sum_{w=n_0}^{n} w^{m-1} Q_j(w) r_{w-\sigma_j(w)} = -\infty.$$

Then every bounded solution of Eq. (1) oscillates.

Proof. Let $\{x_n\}$ be a bounded non-oscillatory solution of Eq. (1). Without loss of generality, we may assume that $x_n > 0$, $x_{n-\tau(n)} > 0$, $x_{n-\sigma_i(n)} > 0$ $(i = 1, 2, \dots, s)$ for $n \geq n_1 \geq n_0$ (the proof when $x_n < 0$, $n \geq n_1$ is similar). By (2) and Lemma 2, there is a $n_2 \geq n_1$, such that

$$(-1)^k \Delta^k z_n > 0 \text{ for } n \geq n_2 \text{ and } k = 0, 1, 2, \dots, m. \tag{7}$$

By Theorem 1, we have $\lim_{n\to\infty} x_n = 0$, and so $\lim_{n\to\infty} x_{n-\sigma_i(n)} = 0$ for $i = 1, 2, \dots, s$. By (c$_3$), there is a $n_3 \geq n_2$, such that as $n \geq n_3$,

$$\frac{f_i(x_{n-\sigma_i(n)})}{x_{n-\sigma_i(n)}} \geq \lambda > 0, \quad \text{for } i = 1, 2, \dots, s. \tag{8}$$

From (1), (2), and (8), we have

$$\Delta^m z_n = \sum_{i=1}^{s} Q_i(n) \frac{f_i(x_{n-\sigma_i(n)})}{x_{n-\sigma_i(n)}} \cdot x_{n-\sigma_i(n)}$$

$$\geq \lambda \sum_{i=1}^{s} Q_i(n) x_{n-\sigma_i(n)} \geq \lambda Q_j(n) x_{n-\sigma_j(n)} \text{ for } n \geq n_3\,. \tag{9}$$

From (2), we have

$$z_n + r_n = x_n - p_n x_{n-\tau(n)} < x_n$$

By (7), we have $z_n > 0$ for $n \geq n_3$. So again, we have $x_n > z_n + r_n > r_n$ for $n \geq n_3$. Hence, we take a $n_4 \geq n_3$, such that

$$x_{n-\sigma_j(n)} > r_{n-\sigma_j(n)} \quad \text{for } n \geq n_4\,. \tag{10}$$

From (9) and (10), we have

$$\Delta^m z_n \geq \lambda Q_j(n) r_{n-\sigma_j(n)} \quad \text{for } n \geq n_4\,. \tag{11}$$

We multiply both sides of (11) by n^{m-1} and then, summing it from n_4 to n, we obtain

$$F_n - F_{n_4} \geq \lambda \sum_{w=n_4}^{n} w^{m-1} Q_j(w) r_{w-\sigma_j(w)}, \tag{12}$$

where

$$F_n = n^{m-1} \Delta^{m-1} z_n - \sum_{l=2}^{m} (-1)^l (m-1)(m-2)\cdots(m-l+1) n^{m-l} \Delta^{m-1} z_n\,.$$

By (7), $F_n < 0$ for $n \geq n_4$. Hence, it follows from (12) that

$$\sum_{w=n_0}^{n} w^{m-1} Q_j(w) r_{w-\sigma_j(w)} \leq -\frac{F_{n_4}}{\lambda},$$

so that

$$\limsup_{n\to\infty} \sum_{w=n_0}^{n} w^{m-1} Q_j(w) r_{w-\sigma_j(w)} \leq -\frac{F_{n_4}}{\lambda}.$$

This contradicts condition (c$_4$), and the proof is complete. ■

Theorem 3. *Let conditions (c_1), (c_3) be satisfied. Assume that the following conditions hold:*

(c_6) *There exists at least an integer j, $1 \leq j \leq s$, such that $\sigma_j(n)$ is non-increasing and as $n \geq n_0$, $\sigma_j(n) > 0$. Moreover, there exists integer sequences $\{n_k\}$ and $\{n'_k\}$, with $\lim_{k\to\infty} n_k = \infty$, $\lim_{k\to\infty} n'_k = \infty$, such that*

$$\begin{cases} \sum_{w=n_k-\sigma_j(n_k)}^{n_k} (n_k - w)^{m-1} Q_j(w) > \frac{(m-1)!}{\lambda}, \\ \sum_{w=n_k-\sigma_j(n_k)}^{n_k} Q_j(w) r_{w-\sigma_j(w)} \geq 0. \end{cases} \tag{H_1}$$

$$\begin{cases} \sum_{w=n'_k-\sigma_j(n'_k)}^{n'_k} (n'_k - w)^{m-1} Q_j(w) > \frac{(m-1)!}{\lambda}, \\ \sum_{w=n'_k-\sigma_j(n'_k)}^{n'_k} Q_j(w) r_{w-\sigma_j(w)} \leq 0. \end{cases} \tag{H_2}$$

Then every bounded solution of Eq. (1) oscillates.

Proof. Let $\{x_n\}$ be a bounded non-oscillatory solution of Eq. (1). Without loss of generality, we may assume that $x_n > 0$, $x_{n-\tau(n)} > 0$, $x_{n-\sigma_i(n)} > 0$ $(i = 1, 2, \ldots, s)$ for $n \geq n_1 \geq n_0$ (the proof when $x_n < 0$ is similar). Observe that $\sigma_j(n)$ is non-increasing, so that there exist a constant $\sigma > 0$ and a $T \geq n_0$, such that $0 < \sigma_j(n) \leq \sigma$ for $n \geq T$, and so $n_k - \sigma \leq n_k - \sigma_j(n_k) \leq w \leq n_k$ for $n_k \geq T$. Hence, we obtain

$$\sum_{w=n_k-\sigma_j(n_k)}^{n_k} (n_k - w)^{m-1} Q_j(w) \leq \sigma^{m-1} \sum_{w=n_k-\sigma_j(n_k)}^{n_k} Q_j(w).$$

By (H_1), we have

$$\sum_{w=n_k-\sigma_j(n_k)}^{n_k} Q_j(w) > \frac{(m-1)!}{\lambda\sigma^{m-1}} > 0.$$

It follows that

$$\sum_{w=n_0}^{\infty} Q_j(w) = \infty.$$

As in the proof of Theorem 2, we have that (7)–(9) hold for $n \geq n_3$. By (2), we have $x_n > z_n + r_n$ for $n \geq n_3$, and so we take a $n_4 \geq n_3$, such that

$$x_{n-\sigma_j(n)} > z_{n-\sigma_j(n)} + r_{n-\sigma_j(n)} \text{ for } n \geq n_4.$$

Combining (9) with the last inequality as $n \geq n_4$, we have

$$\Delta^m z_n \geq \lambda Q_j(n)[z_{n-\sigma_j(n)} + r_{n-\sigma_j(n)}] = \lambda Q_j(n) z_{n-\sigma_j(n)} + \lambda Q_j(n) r_{n-\sigma_j(n)}. \tag{13}$$

By discrete Taylor's formulas [1], set $n_4 \le w \le n_k$. From (13), we have

$$\Delta^m z_w \ge \lambda Q_j(w)[\sum_{i=0}^{m-1} \frac{\Delta^i z_{n_k-\sigma_j(n_k)}}{i!}(w-\sigma_j(w)-n_k+\sigma_j(n_k))^{(i)} + \frac{1}{(m-1)!}\sum_{l=n_k-\sigma_j(n_k)}^{w-\sigma_j(w)-m}(w-\sigma_j(w)-l-1)^{(m-1)}\Delta^m z_l] + \lambda Q_j(w) r_{w-\sigma_j(w)}\,. \tag{14}$$

From (7) and (14), and observe that $\sigma_j(n)$ is non-increasing, we obtain

$$\Delta^m z_w \ge \frac{\lambda}{(m-1)!}\Delta^{m-1} z_{n_k-\sigma_j(n_k)}(n_k-w)^{(m-1)}Q_j(w) + \lambda Q_j(w) r_{w-\sigma_j(w)}. \tag{15}$$

By summing (15) from $n_k - \sigma_j(n_k)$ to $n_k - 1$, we obtain

$$\Delta^{m-1} z_{n_k} - \Delta^{m-1} z_{n_k-\sigma_j(n_k)} \ge \frac{-\lambda}{(m-1)!}\Delta^{m-1} z_{n_k-\sigma_j(n_k)}\sum_{w=n_k-\sigma_j(n_k)}^{n_k-1}(n_k-w)^{m-1}Q_j(w) + \lambda\sum_{w=n_k-\sigma_j(n_k)}^{n_k-1} Q_j(w) r_{w-\sigma_j(w)}\,.$$

Applying (H_1) to the above inequality, we have

$$\Delta^{m-1} z_{n_k} - \Delta^{m-1} z_{n_k-\sigma_j(n_k)} \ge -\Delta^{m-1} z_{n_k-\sigma_j(n_k)},$$

so that $\Delta^{m-1} z_{n_k} \ge 0$ for $n_k \ge n_4$, which contradicts (7), and the proof is complete. ■

Theorem 4. *Let condition (c_1) in Theorem 1 be replaced by*
(c_1') $1 < B_1 \le p_n \le B$ for $n \ge n_0$ where B_1 and B are two positive constants.
Then every bounded non-oscillatory solution of Eq. (1) tends to be zero as $n \to \infty$.

Proof. Let $\{x_n\}$ be a bounded non-oscillatory solution (1). Without loss of generality, we assume that $\{x_n\}$ is eventually positive (the proof when $\{x_n\}$ is eventually negative is similar). By Lemma 2, we have $\lim_{n\to\infty} z_n = 0$. Since $\lim_{n\to\infty} r_n = 0$, therefore, $\lim_{n\to\infty}[z_n + r_n] = 0$. Observe that $\{x_n\}$ is bounded; we may set $\lim_{n\to\infty}\sup x_n = a$ $(0 \le a < \infty)$. We wish to show that $a = 0$. Otherwise, if $a > 0$, then there exists an integer sequence $\{n_k\}$ with $\lim_{k\to\infty} n_k = \infty$, such that $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty}\sup x_n = a > 0$. Since $\{x_n\}$ is bounded, $\{x_{n_k+\tau(n_k)}\}$ is also bounded. Then there is a $\{n_{k_i}\} \subset \{n_k\}$ with $\lim_{i\to\infty} n_{k_i} = \infty$ and such that $\lim_{i\to\infty} x_{n_{k_i}+\tau(n_{k_i})}$ exists. From (2), we have

$$z_{n_{k_i}+\tau(n_{k_i})} + r_{n_{k_i}+\tau(n_{k_i})} \le x_{n_{k_i}+\tau(n_{k_i})} - B_1 x_{n_{k_i}}\,.$$

Letting $i \to \infty$, we have $\lim_{i\to\infty} x_{n_{ki}+\tau(n_{k_i})} \ge B_1 a > a$, which contradicts $\lim_{n\to\infty}\sup x_n = a$. Hence, $a = 0$ holds. Observe that $x_n > 0$ eventually, hence, $\lim_{n\to\infty} x_n = 0$. The proof is complete. ■

Using Lemma 2 and Theorem 4, and following the proof of Theorems 2 and 3, we have

Theorem 5. *Let condition (c_1) in Theorem 2 be replaced by (c_1'). Then every bounded solution of Eq. (1) oscillates.*

Theorem 6. *Let condition (c_1) in Theorem 3 be replaced by (c_1'). Then every bounded solution of Equation (1) oscillates.*

Example. Consider the equation

$$\Delta^4\Big(x_n - \frac{1}{2}x_{n-1}\Big) - 24x_{n-2} = 0, \quad n = 0, 1, 2, \dots . \tag{16}$$

Hence, $m = 4$, $p_n = 1/2$, $r(n) = 1$. $\sigma(n) = 2$, $Q(n) = 24$, and $f(u) = u$. It is easy to verify that the conditions of Theorem 2 are satisfied. Therefore, (16) has an oscillatory solution. For instance, $\{x_n\} = (-1)^n$ is such a solution.

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