Vietnam Journal of Mathematics **27**:3 (1999) 253–258

Vietnam Journal of MATHEMATICS

On the Quasi-Normality and Subnormality of Subgroups of Finite Groups

Akbar Hassani[1] **and Shaban Sedghi**[2]

[1] *Department of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16844, Iran*

[2] *Department of Mathematics, Ghaemshahr Islamic Azad, University Ghaemshahr, Mazandaran, Iran*

Received February 14, 1998
Revised December 8, 1998

Abstract. A subgroup H of a group G is called a *quasi-normal subgroup* of G if $HK = KH$ for all subgroups K of G. We will show that, if H is a quasi-normal subgroup of a group G such that $[G : H]$ is a prime, or $[G : H] = 2^r m$, where $r = 1, 2$, and m is an odd square free number, then H is a normal subgroup of G. However, for an odd prime p and $n \geq 3$ or for $p = 2$ and $n \geq 4$, let G be the group of order p^n with generators a and b, and $a^{p^{n-1}} = 1$, $b^p = 1$, and $ba = a^{1+p^{n-2}}b$. Let $H = \langle b \rangle$. Then $[G : H] = p^{n-1}$ and H is a quasi-normal in G but not normal in G.

1. Introduction

If G is a group and A, B are subgroups of G, the subgroup $\langle A, B\rangle$ of G generated by $A \cup B$ is of interest. To be able to control the properties of the group $\langle A, B\rangle$ to be those of A and B, the generators of $\langle A, B\rangle$ must happen in a special way. The most transparent case we have is when $\langle A, B\rangle$ coincides with the product set $AB = \{ab \mid a \in A,\ b \in B\}$. It is well known that this holds if and only if $AB = BA$. Two subgroups A and B of a group G which have this property are called permutable. A sufficient condition for the permutability of A and B is that A normalizes B (that is, $a^{-1}ba \in B$ for all $a \in A$, $b \in B$) or vice versa. Particularly, if A is a normal subgroup of G, we have $AB = BA = \langle A, B\rangle$ for every subgroup B of G.

In 1939, Ore introduced the concept of a quasi-normal subgroup of a group, which is a generalization of a normal subgroup [5, 13.2.1].

Definition 1. *A subgroup K of G is called a quasi-normal subgroup of G if $HK = KH$ for all subgroups K of G.*

Remark. If H is a subgroup of G, then the following conditions are equivalent:

(i) H is quasi-normal in G;

(ii) For every $g \in G$ and $h \in H$, there exist $r \in Z$ and $h' \in H$ such that $hg = g^r h'$.

We note that $G = \langle x, y \mid x^8 = y^2 = 1,\ y^{-1}xy = x^5 \rangle$ is an example of a group having a quasi-normal subgroup which is not normal, Q_8 is an example of a group having a quasi-normal subgroup which is normal, but D_8 is an example of a group which has a subgroup of index 4 which is not quasi-normal.

The following lemma shows the relation between quasi-normal subgroups and factor groups of normal subgroup contained in such subgroups.

Lemma 1. *If G is a group and $N \subset H \subset G$ are subgroups with N normal in G, then H is quasi-normal in G if and only if H/N is quasi-normal in G/N.*

Proof. It immediately follows from definition. ∎

Of course, every normal subgroup is quasi-normal, which might lead one to hope that subnormal subgroup also have this property. However, the converse is not necessarily true.

One may adopt the opposite point of view, asking whether quasi-normal subgroups are subnormal.

Ore in [5, 13.2.2] shows that if H is a quasi-normal subgroup of a finite group G, then H is subnormal, while, in gereral, a quasi-normal subgroup of an infinite group needs not be subnormal.

Finally, Stonehewer in [6] shows that a quasi-normal subgroup of a finitely generated group G is subnormal.

In 1962, Ito and Szep [4] obtained an interesting result which showed that the difference between normality, in general, is small.

Also, if H is a quasi-normal subgroup of G, then the quotient group H/H_G is nilpotent, that is, H/H_G is contained in the Fitting subgroup $F(G/H_G)$ of G/H_G. Here, H_G denotes the intersection of all conjugates $H^g = g^{-1}Hg$ of H with $g \in G$.

2. When Quasi-Normality Implies Normality

Next, the following theorems show a condition when quasi-normality implies normality.

Theorem 1. *Let H be a quasi-normal subgroup of a group G such that $[G : H]$ is a prime number. Then H is a normal subgroup of G.*

Proof. Suppose this is false. Then there is a conjugate $H' = g^{-1}Hg$ of H such that $H' \neq H$. Let $K = HH' = H'H$. Since $[G : H]$ is prime and $H \subset K \subset G$, $K = G$. In particular, $g = hh'$ for some $h \in H, h' \in H'$. Hence, $g = hg^{-1}h_1g$ for some $h, h_1 \in H$. However, this implies that $g \in H$ so $H' = H$ contradicting the assumption. This completes the proof. ∎

We know that, if the index H in G is equal to 2, then H is normal in G. In the next theorem, we show that a quasi-normal subgroup H of G such that $[G : H] = 4$ is a normal subgroup of G.

Theorem 2. *A quasi-normal subgroup H of G such that $[G : H] = 4$ is a normal subgroup of G.*

Proof. Suppose this is false. Then there is a conjugate $H' = g^{-1}Hg$ of H such that $H' \neq H$. Let $K = HH' = H'H$. Since $H \subset K \subset G$ and $[G : H] = 4$, it follows that K is H or G or else $[K : H] = 2$. If $K = H$, then $H' \subseteq K = H$, so $H' = H$, a contradiction. If $K = G$, then, as in the proof of Theorem 1, H is normal in G. Thus, $[K : H] = 2$ and H is normal in K; also $[G : K] = 2$, and K is normal in G.

We conclude that there are exactly two conjugates of H, namely, H and H'.

Let $N = H \cap H'$. By definition, N is the core of H in G and therefore is a normal subgroup of G. Moreover,

$$[K : H] = [HH' : H] = [H' : N] = [H : N] = 2.$$

Since $N \subset H \subset G$, $[G : H] = 4$, $[H : N] = 2$, and N is normal in G, the group G/N has order 8, H/N is quasi-normal in G/N and has index 4. We know that every quasi-normal subgroup G of order 8 is normal in G. Thus, H/N is normal in G/N, so H is normal in G, contradicting the inital assumption. From this, it follows that H is normal in G. ∎

In general, we show that

Theorem 3. *If H is a quasi-normal subgroup of a group G and $[G : H] = 2^r m$, where $r = 1, 2$, and m is an odd square free number, then H is a normal subgroup of G.*

Proof. We will argue by induction on $n = 2^r m$. If $n = 1$, the result is obvious. For $n = 2^r$, where $r = 1, 2$, it follows from Theorems 1 and 2. Consider any element $g \in G$. Since H is quasi-normal in G, $H\langle g\rangle$ is a subgroup of G and H is quasi-normal in $H\langle g\rangle$. If $H\langle g\rangle \neq G$, then, by induction hypothesis, H is normal in $H\langle g\rangle$, so $Hg = gH$.

If $H\langle g\rangle = G$, then $[H\langle g\rangle : H] = n$. This implies that n is the least positive integer k such that $g^k \in H$. Let $x = g^p$ and $y = g^{m_1}$, where p is prime ($p \neq 2$) and does not divide m_1 ($m_1 = n/p$). Then the least positive integer k such that $x^k \in H$ is $n/p = m_1$. So $[H\langle x\rangle : H] = m_1$. Similarly, $[H\langle y\rangle : H] = p$. Since H is quasi-normal in both $H\langle x\rangle$ and $H\langle y\rangle$, the inductive hypothesis shows that $Hx = xH$ and $Hy = yH$. The fact that $(p, m_1) = 1$ implies that $g \in \langle x, y\rangle$, hence, $Hg = gH$. ∎

Lemma 2. *Let H be a quasi-normal subgroup of a finite group G. If $(n, |G|) = 1$, then H is quasi-normal in the group $G \times Z_n$, where Z_n denotes the cyclic group of order n.*

Proof. Let $k \in G \times Z_n$ and $h \in H$. We will show that $hk = k^{r'}h'$ for some integer r' and $h' \in H$. We have $k = (g, a^s)$ for some $g \in G$ and integer s, where $\langle a\rangle = Z_n$. Since H is quasi-normal in G, $hg = g^r h'$ for some integer r and $h' \in H$. Since $(n, |G|) = 1$, there is an integer r' such that $r' \equiv r \pmod{|G|}$ and $r' \equiv 1 \pmod{n}$. Hence,

$$\begin{aligned} hk &= (h, 1)(g, a^s) = (hg, a^s) = (g^r h', a^s) = (g^{r'} h', a^s) \\ &= (g^{r'}, a^s)(h', 1) = (g^{r'}, a^{r's})(h', 1) = (g, a^s)^{r'}(h', 1) = k^{r'}h' \end{aligned}$$

and H is quasi-normal in $G \times Z_n$. ∎

3. When Quasi-Normal Does Not Imply Normal

For any positive integer m that is divisble by 8 or the square of an odd prime, we will exhibit a finite group G and a quasi-normal subgroup H such that $[G : H] = m$ and H is not normal in G.

Given a group G and $a, b \in G$, let $[a, b]$ denote $a^{-1}b^{-1}ab$, the commutator of a and b. Then we have [2, Lemma 2.2].

(i) If $[a, b]$ commutes with a, then $[a^n, b] = [a, b]^n$ for any $n \in Z$.
(ii) If $[a, b]$ commutes with a and b, then, for any integer $n \geq 0$,

$$(ab)^n = a^n b^n [b, a]^{\binom{n}{2}}.$$

Lemma 3. *For an odd prime p and $n \geq 3$ or for $p = 2$ and $n \geq 4$, let G be the group of order p^n with generators a and b and $a^{p^{n-1}} = 1$, $b^p = 1$, and $ba = a^{1+p^{n-2}}b$. Let $H = \langle b \rangle$. Then $[G : H] = p^{n-1}$ and H is quasi-normal but not normal in G.*

Proof. Every element in G has a unique representation in the form $a^i b^j$ with $0 \leq i < p^{n-1}$, $0 \leq j < p$. Since $a^{-1}ba = a^{p^{n-2}}b \notin H$, H is not normal in G. To show that H is quasi-normal in G, we first note the following:

Since $ba^pb^{-1} = (bab^{-1})^p = (a^{1+p^{n-2}})^p = a^{p+p^{n-1}} = a^p$, we have $a^p \in Z(G)$ and $a^{p^{n-2}} \in Z(G)$. Since

$$[b, a] = b^{-1}(a^{-1}ba) = b^{-1}a^{p^{n-2}}b = a^{p^{n-2}},$$

we have $[b, a]$, $[a, b] \in Z(G)$. So, for any $i, j \in Z$, $[b^j, a^i] = [b, a]^{ij} \in Z(G)$ by (i), and similarly, $[a^i, b^j] \in Z(G)$. Also,

$$[b, a]^p = (a^{p^{n-2}})^p = a^{p^{n-1}} = 1.$$

Let $g \in G$ and $h \in H$. Then $g = a^ib^j$ and $h = b^k$ for some $i, j, k \geq 0$. Let $r = 1 + p^{n-2}k$. By the remark after Definition 1, it suffices to show that $hg = g^rh$. By (ii),

$$g^r = (a^ib^j)^r = a^{ir}b^{jr}[b^j, a^i]^{\binom{r}{2}}.$$

Note that

$$a^{ir} = a^i(a^{p^{n-2}})^{ik} = a^i[b, a]^{ik} = a^i[b^k, a^i],$$

and that

$$b^{jr} = b^{j+p^{n-2}jk} = b^j.$$

Also, the restrictions on p and n imply that p divides $\binom{r}{2}$, and $[b^j, a^i]^{\binom{r}{2}} = [b, a]^{ij\binom{r}{2}} = 1$, since $[b, a]^p = 1$. Thus, $g^r = a^i[b^k, a^i]b^j$. Consequently,

$$\begin{aligned} g^rh = g^rb^k &= a^i[b^k, a^i]b^{j+k} = a^ib^k[b^k, a^i]b^j \\ &= a^ib^k(b^{-k}a^{-i}b^ka^i)b^j = b^ka^ib^j = hg. \end{aligned}$$

■

4. On Some Products of Conjugate-Permutable Subgroups

In the proof that a quasi-normal subgroup is subnormal [5], one only needs to show that it is permutable with all of its conjugates. This leads to a new concept as follows:

Definition 2. *A subgroup H of a group G is called a conjugate-permutable subgroup of G ($H <_{c-p} G$) if $HH^g = H^g H$ for all $g \in G$.*

In this section we prove that conjugate-permutable subgroups are subnormal, and we prove some elementary properties of conjugate-permutable subgroups. We also give examples of subnormal subgroups which are not conjugate-permutable subgroups, and of conjugate-permutable subgroups that are not quasi-normal.

Of course, every quasi-normal subgroup is a conjugate-permutable subgroup. However, the converse is not necessarily true.

Example. We note that $H = \langle yx \rangle$ is a conjugate-permutable subgroup of $D_8 = \langle x, y \mid x^4 = y^2 = 1,\ y^{-1}xy = x^{-1} \rangle$, but H is not a quasi-normal subgroup of D_8.

As in the proof of Theorem 1, it is easy to see that, if H is a conjugate-permutable subgroup of a group G such that $[G : H]$ is a prime number, then H is a normal subgroup of G. Also, if H is a maximal conjugate-permutable subgroup of G, then H is a normal subgroup of G.

Corollary 1. *If $H <_{c-p} G$ and G is finite group, then H is subnormal.*

Example. Let $D_{16} = \langle x, y \mid x^8 = y^2 = 1,\ y^{-1}xy = x^{-1} \rangle$, $H = \langle y \rangle$, and $K = \langle yx^6 \rangle$. Then H is subnormal in D_{16} (since D_{16} is nilpotent), but

$$HK = \{1, yx^6, y, x^6\} \neq \{1, yx^6, y, x^2\} = KH.$$

So H is not a conjugate-permutable subgroup.

Corollary 2. *If G is a finite group with all maximal subgroups conjugate-permutable, then G is nilpotent.*

Foguel in [1] proved the following theorem: If G is a finite group and there exist $H <_{c-p} G$ such that H is a maximal subgroup of a $P \in Syl_2(G)$, then G is solvable.

Huppert [3, Satz 10.3] proved the following theorem: If a finite group is the product of pairwise permutable cyclic subgroups, then it is supersolvable. Of course the converse of this statement is not even true in the class of nilpotent groups as shown in the above example.

Assume G is a finite group. $\pi(G)$ denotes the set of prime divisors of the order of the group G.

Lemma 4. *Let P be a normal p-subgroup, Q a Sylow q-subgroup of a group G, $p \neq q$, and H a subgroup of P such that $HQ = QH$. Then H is normalized by Q.*

Proof. It is easy. ■

Theorem 4. *Let H be an abelian normal subgroup of a G such that $[G, G] \leq H$ and the Sylow subgroups of H are elementary abelian. Assume that, for every $q \in \pi(H)$, the Sylow q-subgroup Q of H can be written as $Q = Q_1 \cdots Q_s$, where Q_i is a cyclic and permutable with Sylow p-subgroups of G for all $p \in \pi(G)$ and $1 \leq i \leq s$. Then G is supersolvable.*

Proof. We prove the claim by induction on the order of G. We show that Q contains a normal subgroup of order q of G. Let Q^* be a Sylow q-subgroup of G. Then $Q \leq Q^*$. Let $1 = B_0 \lhd B_1 \lhd \cdots \lhd B_r = Q$ such that $B_i \lhd Q^*$ and B_i/B_{i-1} is of order q for all i. Let $1 \leq t \leq r$ be minimal such that B_t contains a subgroup A of G such that A is permutable with Sylow p-subgroups of G for all $p \in \pi(G)$, where A may be taken to be one of Q_i's. If A is normal in Q^*, then since H is an abelian normal subgroup of G and Q is a Sylow q-subgroup of H, it is easy to show that Q is a normal subgroup of G. Since, for every $x \in Q$ and every $b \in G$, we have $x^b \in H$ and $|x| = |x^b|$, hence x^b is a q-element, so it belongs to only Sylow q-subgroups Q of H. Since A is permutable with Sylow p-subgroups of G for all $p \in \pi(G)$, hence, by Lemma 4, A is a normal subgroup in all Sylow p-subgroups of G so A is normal in G.

Let $x \in P$. Then $a^x = a^t$ for some integer t, hence, $(a^b)^{x^b} = (a^b)^t$. For $(a^b)^t = (a^t)^b = (a^x)^b = a^{xb} = (a^b)^{x^b}$. As $[G, G] \leq H, a, a^b \in H$ and $a_1 = a^b$, we obtain that

$$a_1{}^x = (a^b)^x = (a^x)^{b^x} = (a^t)^{b^x} = (a^t)^{bb^{-1}x^{-1}bx}$$
$$= ((a^t)^b)^{[b,x]} = ((a^b)^t)^{[b,x]} = (a_1)^t.$$

Hence, $(a_1)^x = (a_1)^t$ for every $x \in P$. So, by similar arguments as above for $A^b = \langle a_1 \rangle$, it follows that

$$(a_1{}^b)^t = (a_1{}^b)^{x^b} = (a_1{}^b)^{x[x,b]} = ((a_1{}^b)^x)^{[x,b]} = (a_1{}^b)^x.$$

Hence, every element of P acts on $\langle a, a_1{}^b \rangle$ by raising to some power t.

Now, we show that $\langle aa_1{}^{-b} \rangle$ is permutable with Sylow p-subgroups. Since $aB_{t-1} = (aB_{t-1})^b = a^b B_{t-1} = a_1 B_{t-1} = (a_1 B_{t-1})^b = a_1{}^b B_{t-1}$, it follows that $aa_1{}^{-b} \in B_{t-1}$.

Now, it follows that $\langle aa_1{}^{-b} \rangle$ is permutable with Sylow p-subgroups of G for all $p \in \pi(G)$ and $\langle aa_1{}^{-b} \rangle$ is contained in B_{t-1}, a contradiction. Thus, A is normal in G. It is easy to see that G/A satisfies the conditions of our theorem. Consequently, G/A is supersolvable, which implies the supersolvability of G. ■

References

1. T. Foguel, Conjugate-permutable, *J. Algebra* **191** (1997) 235–239.
2. D. Gorenstein, *Finite Groups*, Harper's Series in Modern Mathematics, Harper & Row, New York, 1968.
3. B. Huppert, *Endlichen Gruppen I*, Springer-Verlag, New York, 1967.
4. N. Ito and J. Szep, Über die Quasinormalteiler von endlicher Gruppen, *Acta Sci. Math.* **23** (1962) 168–170.
5. D.J.S. Robinson, *A Course in the Theory of Groups*, 2nd ed., Springer-Verlag, New York, 1996.
6. S.E. Stonehewer, Permutable subgroups of infinite groups, *Math. Z.* **125** (1972) 1–16.