Vietnam Journal of Mathematics **27**:3 (1999) 273–275

Vietnam Journal
of
MATHEMATICS

Short Communication

An Alternative Approach to the Associative Calibration

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Received October 5, 1998
Revised April 2, 1999

The associative calibration is a well-known 3-calibration on $\mathbf{R}^7 \cong \mathbf{ImO}$, defined to depend on the octonionic structure.

Suppose φ is a 3-vector and $\{e_1, e_2, \ldots, e_7\}$ is the canonical orthonormal basis on $\mathbf{R}^7$. Usually, φ is given by an expression in terms of axis 3-planes

$$\varphi = \sum_{i<j<k} a_{ijk} e_i^* \wedge e_j^* \wedge e_k^*.$$

We are interested in the following question: How does one know whether or not the calibration is associative?

The associative calibration is always determined and recognized by its canonical form. So the question is to find a suitable orthonormal basis, where φ can be expressed in the simplest form. From this, we will know whether φ is the associative calibration or not.

This note gives an answer to this question based on the investigation of the C-algebra $(L, [.,.])$ associated with φ (see [7]).

φ is the associative calibration if and only if $(L, [.,.])$ satisfies the following condition:

$$ad_w^2 = -|w|^2 Id_{w^\perp} \quad \forall w \in L, \tag{$*$}$$

i.e.,

$$[w, [w, x]] = -|w|^2 x \quad \forall x \in w^\perp.$$

Our main results are the following.

Theorem 1. *Let $(L, [.,.])$ be a non-commutative C-algebra, J the Jacobiator on L, and $x, y, z \in L$ orthogonal vectors. If $ad_w^2 = -|w|^2 Id_{w^\perp}$ $\forall w \in L$, then*

(1) *ad_x, ad_y are anti-commutative on $(x, y)^\perp$, i.e., $ad_x(ad_y(z)) = -ad_y(ad_x(z))$ for all $z \in L$, where $(x, y)^\perp$ is the subspace of L containing all vectors, which are orthogonal to x and y;*

(2) $[x, [y, x]] = -[x, [z, y]] = -[y, [x, z]] = [y, [z, x]] = -[z, [y, x]] = [z, [x, y]]$;

(3) $J(x, y, z) = [x, [y, z]]$;

(4) $|[x, y]| = |x|.|y| = |x \wedge y|$;

(5) Span $(x \wedge y \wedge z)$ *is a 3-dimensional C-subalgebra if and only if* $J(x, y, z) = 0$.

The following theorem gives a criterion to verify whether a C-algebra satisfies the condition $(*)$ or not.

Theorem 2. *Let L be a non-commutative C-algebra, and $\{e_1, e_2, \ldots, e_n\}$ an orthonomal basis of L. If $ad_{e_i}^2 = -Id_{e_i^\perp}$ for all $i = 1, 2, \ldots, n$ and ad_{e_i}, ad_{e_j} are anti-commutative on $(e_i, e_j)^\perp$ for all $i \neq j$, then $ad_w^2 2 = -|w|^2 Id_{w^\perp}$ $\forall w \in L$.*

Let L be a C-algebra satisfying the condition $(*)$. We have

Theorem 3.

$$\langle x, [y, z]\rangle^2 + |J(x, y, z)|^2 = |x \wedge y \wedge z|^2 \quad \forall x, y, z \in L.$$

By Theorem 3, we obtain

Conclusion 4.

(1) The form $\varphi(x, y, z) = \langle x, [y, z]\rangle$ is a calibration, i.e., it has comass 1.

(2) $G(\varphi) \cup G(-\varphi) = G_0(J) = \{\pm[y, z] \wedge y \wedge z \mid y \wedge z \in G(2, L)\}$.

(3) $G_0(\varphi) = G(J)$.

(4) φ is a maximal calibration.

We also prove that $\mathbf{R} \oplus L$ with the operation defined by

$$(a, b).(b, y) = (ab - \langle x, y\rangle, ay + bx + [x, y]),$$

is a normed algebra, and hence, L must be of dimension three or seven.

A direct computation shows that the C-algebras associated with $\varphi = e_1^* \wedge e_2^* \wedge e_3^*$ and the associative calibration satisfy the condition $(*)$.

Conversly, if φ is the 3-vector associated with a 7-dimensional C-algebra, which satisfies $(*)$, then, by Conclusion 4, we can choose a 3-covector $\xi = a_1 \wedge a_2 \wedge a_3 \in G(\varphi)$ (a_1, a_2, a_3 are orthonormal vectors). Let $a_4 \perp a_1, a_2, a_3$ and set

$$[a_1, a_4] = -a_5; \ [a_2, a_4] = -a_6; \ [a_3, a_4] = -a_7.$$

Then, by the condition $(*)$, we can prove that $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ is an orthonormal basis of $\mathbf{R}^7$. This is the required basis so that φ can be recognized as the associative calibration.

In other words, by Theorem 2, we know whether φ is the associative calibration or not, and if it is, then we can choose a suitable orthonormal basis so that φ is in the simplest form (the canonical form). The coassociative and Cayley calibrations can also be investigated in this way.

The results of this note will be published in detail elsewhere.

Acknowledgements. The author would like to thank Professor Dao Trong Thi for many valuable discussions, and Professor Frank Morgan for materials on calibrated geometries.

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