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Outer γ-Convexity in Normed Linear Spaces

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Abstract. In this paper we introduce the notion of outer γ-convex sets and outer γ-convex functions in a normed linear space X, and show, among other things, that a function $f : D \subseteq X \to \mathbb{R}$ is outer γ-convex if and only if the level set $\{x \in D : f(x) + \xi(x) \leq \alpha\}$ is outer γ-convex for every continuous linear functional $\xi \in X^*$ and for every real α. Two main properties of outer γ-convex functions are: (M_γ) each γ-minimizer (defined by $f(x*) \leq f(x)$ for all $x \in D$ satisfying $\|x - x^*\| < \gamma + \varepsilon$ for some $\varepsilon > 0$) is a global minimizer, and (I_γ) each γ-infimizer (defined by $\liminf_{y \to x^*} f(y) \leq f(x)$ for all $x \in D$ satisfying $\|x - x^*\| < \gamma + \varepsilon$ for some $\varepsilon > 0$) is a global infimizer (i.e., $\liminf_{y \to x^*} f(y) \leq f(x)$ for all $x \in D$). Moreover, for $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, if $f + \xi$ fulfills (M_γ) or (I_γ) for all linear functionals ξ on $\mathbb{R}$, then f must be outer γ-convex (if it is in addition lower semi-continuous), or lsc$f$ is outer γ-convex, respectively.

1. Introduction

One of the most important properties of convex functions with respect to optimization is

$$(M) \qquad \text{each local minimizer is a global minimizer.}$$

To obtain this property for a more general class of functions, the explicit quasiconvexity was introduced (see [3]). Explicitly quasiconvex functions possess the property (M), but in general it does not hold if they are disturbed by some linear functionals (even with sufficiently small norm). In this sense, we say that explicitly quasiconvex functions are not *stable* with respect to (M) (see [7]). To get generalized convex functions which are stable with respect to (M), the so-called s-quasiconvexity was defined in [7]. It was showed there that an s-quasiconvex function f is stable with respect to (M), i.e., there is an $\varepsilon > 0$ such that $f + \xi$ possesses (M) for all continuous linear functional ξ satisfying $\|\xi\| < \varepsilon$.

If some property of the function f is required to remain true even if it is disturbed by arbitrary continuous linear functionals, then f is said to be *absolutely stable* with respect to this property (see [8]). Obviously, convex functions are absolutely stable with respect to (M). It is more interesting that among lower semi-continuous functions defined on some compact interval of $\mathbb{R}$, only convex functions can be absolutely stable with respect to (M) (see [8]).

A question we are interested in is: Which functions are absolutely stable with respect to the property

$$(M_\gamma) \qquad \text{each } \gamma\text{-minimizer is a global minimizer?}$$

(See definition in Sec. 4.) It was proved in [8] that lower semi-continuous γ-convex like functions are absolutely stable with respect to (M_γ). Moreover, if a lower semi-continuous function defined on some compact interval of $\mathbb{R}$ is absolutely stable with respect to (M_γ), then it must be γ-convex like.

As we see, to obtain the mentioned result in [8], one needs the assumption of lower semi-continuity. To avoid this, the outer γ-convexity of functions is introduced in Sec. 3. As Proposition 4.2 points out, each outer γ-convex function is absolutely stable with respect to (M_γ), and also to the property

$$(I_\gamma) \qquad \text{each } \gamma\text{-infimizer is a global infimizer}$$

(see definition in Sec. 4). Moreover, if a function $f : [a, b] \to \mathbb{R}$ is bounded from below and is absolutely stable with respect to (I_γ), then its lower semi-continuous hull must be outer γ-convex (Proposition 4.5).

To get a similar relation as in the classical convexity, we define the notion of outer γ-convex sets (Sec. 2) and show that a function is outer γ-convex if and only if it is absolutely stable with respect to the property: each lower level set is outer γ-convex (Proposition 3.2).

2. Outer γ-Convex Sets

Let $(X, \|\cdot\|)$ be a normed linear space and γ a fixed positive real number. For any x_0, $x_1 \in X$ and $\lambda \in [0, 1]$, we denote

$$\begin{aligned} x_\lambda &:= (1-\lambda)x_0 + \lambda x_1, \\ [x_0, x_1] &:= \{x_\lambda : 0 \le \lambda \le 1\}, \\ [x_0, x_1[&:= [x_0, x_1] \setminus \{x_1\}, \\ ]x_0, x_1[&:= [x_0, x_1[\setminus \{x_0\}. \end{aligned} \tag{2.1}$$

Definition 2.1. *A subset $M \subseteq X$ is said to be outer γ-convex if, for all x_0 and x_1 in M, there exist $k \in \mathbb{N}$ and*

$$\begin{aligned} &\lambda_i \in [0, 1], \; i = 0, 1, \ldots, k, \quad \text{with } \lambda_0 = 0, \; \lambda_k = 1, \\ &0 \le \lambda_{i+1} - \lambda_i \le \frac{\gamma}{\|x_0 - x_1\|} \text{ for } i = 0, 1, \ldots, k-1, \end{aligned} \tag{2.2}$$

such that

$$x_{\lambda_i} \in M \text{ for } i = 0, 1, \ldots, k. \tag{2.3}$$

Observe that if $\|x_0 - x_1\| \le \gamma$, then the above condition is always fulfilled for $k = 1$.

Due to (2.1), the conditions in (2.2)–(2.3) mean that $x_{\lambda_0} = x_0$, $x_{\lambda_k} = x_1$, $x_{\lambda_i} \in M \cap [x_0, x_1]$, and

$$\|x_{\lambda_i} - x_{\lambda_{i+1}}\| = (\lambda_{i+1} - \lambda_i)\,\|x_0 - x_1\| \le \gamma \quad \text{for } i = 0, 1, \ldots, k-1. \tag{2.4}$$

The reason why we call such a set M outer γ-convex is that a segment connecting two points of M may contain at most gaps (i.e., subsegments outside M) whose length is smaller than γ. More precisely,

Proposition 2.1. *Let $M \subset X$ be outer γ-convex, and let x_0 and x_1 belong to M. Then*

$$[x_0', x_1'[\subset [x_0, x_1] \setminus M \quad \textit{implies} \quad \|x_0' - x_1'\| < \gamma.$$

Proof. By definition and (2.4), there exist $k \in \mathbb{N}$ and $x_{\lambda_i} \in M \cap [x_0, x_1]$ with $x_{\lambda_0} = x_0$, $x_{\lambda_k} = x_1$, and $\|x_{\lambda_i} - x_{\lambda_{i+1}}\| \le \gamma$ for $i = 0, 1, \ldots, k-1$. Since $[x_0', x_1'[\subset [x_0, x_1] \setminus M$, there is some $j \in \{0, 1, \ldots, k-1\}$ such that $x_0' \in]x_j, x_{j+1}[$ and $x_1' \in [x_j, x_{j+1}]$. Therefore, $\|x_0' - x_1'\| < \|x_j - x_{j+1}\| \le \gamma$. ■

The number k mentioned in Definition 2.1 can be very large and it possibly tends to infinity during some convergence consideration. To avoid this negative effect, we use the following assertion.

Proposition 2.2. *Every set $\{\lambda_0, \lambda_1, \ldots, \lambda_k\}$ satisfying (2.2) contains a subset which has at most*

$$\bar{k} := 2\,\mathrm{rd}\left(\frac{\|x_0 - x_1\|}{\gamma}\right) - 1 \tag{2.5}$$

elements and also fulfills (2.2), where

$$\mathrm{rd}(x) := \min\{z \in \mathbb{Z} : z \ge x\}. \tag{2.6}$$

Proof. Assume that $k > \bar{k}$. With

$$j^* := \mathrm{rd}\left(\frac{\|x_0 - x_1\|}{\gamma}\right), \tag{2.7}$$

define

$$\beta_j := \frac{j\gamma}{\|x_0 - x_1\|}, \quad 0 \le j \le j^* - 1, \quad \beta_{j^*} := 1. \tag{2.8}$$

For each $j \in \{1, 2, ..., j^* - 1\}$, $\beta_j - \beta_{j-1} = \gamma / \|x_0 - x_1\|$ and (2.2) imply that there exists some $\lambda_i \in [\beta_{j-1}, \beta_j]$. Therefore, we can determine $i(j) \in \{0, 1, ..., k\}$ such that

$$\lambda_{i(j)} = \max\{\lambda_i : \beta_{j-1} \le \lambda_i \le \beta_j\}.$$

Let $i(0) := 0$ and $i(j^*) := 1$. Due to the definition and (2.2), we have

$$\beta_j \le \lambda_{i(j)+1} \le \lambda_{i(j+1)} \le \beta_{j+1} \quad \text{for } j = 0, 1, ..., j^* - 1.$$

Therefore, it follows from

$$\mu_0 := \lambda_0 = 0,\ \mu_{2j^*-1} := \lambda_1 = 1,\ \mu_{2j-1} := \lambda_{i(j)},\ \mu_{2j} = \lambda_{i(j)+1},\ j = 1, 2, ..., j^*-1,$$

and $0 \le 1 - \beta_{j^*-1} \le \gamma / \|x_0 - x_1\|$ (due to (2.6)–(2.8)) that

$$0 \le \mu_{i+1} - \mu_i \le \frac{\gamma}{\|x_0 - x_1\|} \quad \text{for } i = 0, 1, \ldots, 2j^* - 2.$$

Since $\bar{k} = 2j^* - 1$, $\{\mu_i : 0 \le i \le 2j^* - 1\}$ is the subset we have to seek for. ■

Relation (2.4) and Proposition 2.2 immediately yield the following.

Proposition 2.3. *M is outer γ-convex if and only if, for all x_0 and x_1 in M, there exist a natural number k and $x_{\lambda_i} \in M \cap [x_0, x_1]$, $i = 0, 1, \ldots, k$, such that*

$$x_{\lambda_0} = x_0,\ x_{\lambda_k} = x_1,\ \text{and}\ \|x_{\lambda_i} - x_{\lambda_{i+1}}\| \le \gamma \text{ for } i = 0, 1, \ldots, k-1, \tag{2.9}$$

where k can be restricted by $k \le \bar{k} = 2\,\mathrm{rd}\,(\|x_0 - x_1\|/\gamma) - 1$.

The proof of the next assertion presents another application of Proposition 2.2.

Proposition 2.4. *Suppose $M \subset X$. If M is outer γ-convex, then $\mathrm{cl}M$ is outer γ-convex, too.*

Proof. Let x_0^* and x_1^* be two arbitrary points of $\mathrm{cl}M$. Then there are two sequences (x_0^n) and (x_1^n) in M such that

$$\lim_{n\to\infty} x_0^n = x_0^* \quad\text{and}\quad \lim_{n\to\infty} x_1^n = x_1^*. \tag{2.10}$$

Since M is outer γ-convex and $x_0^n, x_1^n \in M$, there exist $k^n \in \mathbb{N}$ and $\lambda_i^n \in [0,1]$, $i = 0, 1, \ldots, k^n$, satisfying $\lambda_0^n = 0$, $\lambda_k^n = 1$,

$$0 \le \lambda_{i+1}^n - \lambda_i^n \le \frac{\gamma}{\|x_0^n - x_1^n\|} \quad \text{for } i = 0, 1, \ldots, k^n - 1, \tag{2.11}$$

and

$$x_{\lambda_i^n}^n = (1 - \lambda_i^n)x_0^n + \lambda_i^n x_1^n \in M \quad \text{for } i = 0, 1, \ldots, k^n. \tag{2.12}$$

Assume, without loss of generality, that $\|x_0^n - x_1^n\| \le \|x_0^* - x_1^*\| + \gamma$ for all $n \in \mathbb{N}$. Due to Proposition 2.2, we can choose

$$k^n \le 2\,\mathrm{rd}\left(\frac{\|x_0^n - x_1^n\|}{\gamma}\right) - 1 \le k^* := 2\,\mathrm{rd}\left(\frac{\|x_0^* - x_1^*\|}{\gamma}\right) + 1, \quad n = 1, 2, \ldots .$$

If $k^n < k^*$, then set $\lambda_i^n = 1$ and $x_{\lambda_i^n}^n = x_1^n$ for $k^n < i \le k^*$. In such a way, we can obtain $k^n = k^*$ for all $n \in \mathbb{N}$. The tuples $(\lambda_1^n, \lambda_2^n, \ldots, \lambda_{k^*}^n)$, $n \in \mathbb{N}$, form a bounded sequence in $\mathbb{R}^{k^*}$. Therefore, we can assume, without loss of generality that

$$\lim_{n\to\infty} \lambda_i^n = \lambda_i^*, \quad i = 0, 1, \ldots, k^*. \tag{2.13}$$

(2.11) and (2.13) imply $0 = \lambda_0^* \le \lambda_1^* \le \cdots \le \lambda_{k^*}^* = 1$ and

$$\lambda_{i+1}^* - \lambda_i^* \le \lim_{n\to\infty} \frac{\gamma}{\|x_0^n - x_1^n\|} = \frac{\gamma}{\|x_0^* - x_1^*\|} \quad \text{for } i = 0, 1, \ldots, k^* - 1.$$

Finally, due to (2.10) and (2.12),

$$x_{\lambda_i^*}^* = (1 - \lambda_i^*)x_0^* + \lambda_i^* x_1^* = \lim_{n\to\infty} \left((1 - \lambda_i^n)x_0^n + \lambda_i^n x_1^n\right) = \lim_{n\to\infty} x_{\lambda_i^n}^n,$$

which yields that $x_{\lambda_i^*}^* \in \mathrm{cl}M$ for $i = 0, 1, \ldots, k^*$. Hence, $\mathrm{cl}M$ is outer γ-convex. ■

Note that if M is contained in some convex set $D \subset X$, then the same proof shows that if M is outer γ-convex, so is the relative closure of M in D (i.e., $D \cap \mathrm{cl}M$), too.

Let us now introduce a weaker notion of generalized convexity.

Definition 2.2. *A subset $M \subseteq X$ is said to be γ-convex like if $]x_0, x_1[\, \cap M \neq \emptyset$ holds true for all x_0 and x_1 in M satisfying $\|x_0 - x_1\| > \gamma$.*

Clearly, each convex-like set is γ-convex like (for an arbitrary $\gamma \geq 0$), and each outer γ-convex set is γ-convex like. In general, the converse does not hold. For example, the set $[0, 1[\cup\{2\} \subset \mathbb{R}$ is γ-convex like for an arbitrary $\gamma \geq 0$, but it is not outer γ-convex for $\gamma \leq 1$. Under the closedness assumption, we have the following equivalence.

Proposition 2.5. *Suppose $M \subseteq D \subseteq X$, D is convex and M is relatively closed in D. Then M is outer γ-convex if and only if it is γ-convex like.*

Proof. Clearly, we only have to show that if M is γ-convex-like, then it is outer γ-convex. Without loss of generality, assume $M \subseteq D \subseteq \mathbb{R}$. Let x_0 and x_1 be in M and $x_1 - x_0 > \gamma$. With j^* defined in (2.7), determine

$$y_j := x_0 + j\gamma \frac{x_1 - x_0}{\|x_0 - x_1\|} \quad \text{for } 0 \leq j \leq j^* - 1 \quad \text{and} \quad y_{j^*} := x_1.$$

Since M is closed and $x_0, x_1 \in M$, there exist y_j^- and y_j^+ in $M \cap [x_0, x_1]$ satisfying

$$y_j^- = \max_{x \in [x_0, y_j] \cap M} x \quad \text{and} \quad y_j^+ = \min_{x \in [y_j, x_1] \cap M} x, \quad j = 1, 2, \ldots, j^* - 1.$$

This yields that either $y_j^- = y_j^+ = y_j \in M$, or $y_j \notin M$, $\{y_j^-, y_j^+\} \subset M$ and $]y_j^-, y_j^+[\, \cap M = \emptyset$. Since M is γ-convex like, we have

$$0 \leq y_j^+ - y_j^- \leq \gamma, \quad j = 1, 2, \ldots, j^* - 1, \tag{2.14}$$

which implies immediately $y_j^+ - y_j \leq y_j^+ - y_j^- \leq \gamma$ and $y_{j+1} - y_{j+1}^- \leq y_{j+1}^+ - y_{j+1}^- \leq \gamma$, i.e.,

$$y_j^+, \; y_{j+1}^- \in [y_j, y_{j+1}], \quad j = 1, 2, \ldots, j^* - 1.$$

Consequently, by definition,

$$0 \leq y_{j+1}^- - y_j^+ \leq y_{j+1} - y_j = \gamma, \quad j = 1, 2, \ldots, j^* - 1. \tag{2.15}$$

Moreover, $0 \leq y_1^- - x_0 \leq \gamma$ and $0 \leq x_1 - y_{j^*-1}^+ \leq \gamma$. Consequently, due to (2.14)–(2.15), we obtain with

$$x_{\lambda_0} = x_0, \; x_{\lambda_{2j^*-1}} = x_1, \; x_{\lambda_{2j-1}} := y_j^- \text{ and } x_{\lambda_{2j}} := y_j^+, \; j = 1, 2, \ldots, j^* - 1,$$

a chain $x_{\lambda_i} \in M \cap [x_0, x_1]$, $i = 0, 1, \ldots, k := 2j^* - 1$, which satisfies (2.9). Therefore, by Proposition 2.3, M is outer γ-convex. ∎

Outer γ-convex sets still have other interesting properties, but we will deal with them in another paper. Here, only some results are stated which are useful for the next section.

3. Outer γ-Convex Functions

In this section, we introduce the γ-convexity of functions given on some convex subset D of the normed linear space X.

Definition 3.1. *$f : D \to \mathbb{R} \cup \{-\infty\}$ is called outer γ-convex if, for all x_0 and x_1 in D, there exist $k \in \mathbb{N}$ and $\lambda_i \in [0, 1]$, $i = 0, 1, \ldots, k$, satisfying (2.2) such that*

$$f(x_{\lambda_i}) \le (1 - \lambda_i) f(x_0) + \lambda_i f(x_1) \text{ for } 0 \le i \le k. \qquad (3.1)$$

Observe that if $\|x_0 - x_1\| \le \gamma$, then the above condition is always fulfilled for $k = 1$.

By Proposition 2.2, we can restrict ourselves to $k \le \bar{k} = 2\,\mathrm{rd}(\|x_0 - x_1\|/\gamma) - 1$, i.e., f is outer γ-convex if and only if, for all x_0 and x_1 in D satisfying $\|x_0 - x_1\| > \gamma$, there exist $k \le \bar{k}$ and $\lambda_i \in [0, 1]$, $i = 0, 1, \ldots, k$, such that (2.2) and (3.1) are fulfilled.

Note that almost all types of roughly convex functions introduced in [1, 2, 4, 5, 10] are special kinds of outer γ-convex functions.

The following assertion is obvious.

Proposition 3.1. *The sum of an outer γ-convex function and a convex function is outer γ-convex.*

Similar to convex functions, outer γ-convex functions can be characterized by their lower level sets

$$\mathcal{L}_\alpha(f) := \{x \in D : f(x) \le \alpha\}.$$

Proposition 3.2. *The function $f : D \subset X \to \mathbb{R}$ is outer γ-convex if and only if, for every continuous linear functional ξ on X and for every real number α, the level set $\mathcal{L}_\alpha(f + \xi)$ is outer γ-convex.*

Proof. Necessity. Assume that $\xi \in X^*$, $\alpha \in \mathbb{R}$ and $x_0, x_1 \in \mathcal{L}_\alpha(f + \xi)$ with $\|x_0 - x_1\| > \gamma$. Obviously, $f + \xi$ is outer γ-convex because f is outer γ-convex and ξ is linear. Therefore, by definition, there exist $k \in \mathbb{N}$ and $\lambda_i \in [0, 1]$, $i = 0, 1, \ldots, k$, satisfying (2.2) such that

$$f(x_{\lambda_i}) + \xi(x_{\lambda_i}) \le (1 - \lambda_i)(f(x_0) + \xi(x_0)) + \lambda_i(f(x_1) + \xi(x_1)) \le \alpha,$$

i.e., $x_{\lambda_i} \in \mathcal{L}_\alpha(f + \xi)$ for $0 \le i \le k$. Hence, $\mathcal{L}_\alpha(f + \xi)$ is outer γ-convex.

Sufficiency. Assume $\mathcal{L}_\alpha(f + \xi)$ is outer γ-convex for every $\xi \in X^*$ and $\alpha \in \mathbb{R}$. Let x_0 and x_1 be in D and satisfy $\|x_0 - x_1\| > \gamma$. We now choose $\xi \in X^*$ such that $\xi(x_1 - x_0) = -f(x_1) + f(x_0)$. Since ξ is linear, this yields $f(x_0) + \xi(x_0) = f(x_1) + \xi(x_1)$. Thus, for $\alpha = f(x_0) + \xi(x_0)$, x_0 and x_1 belong to $\mathcal{L}_\alpha(f + \xi)$. By the outer γ-convexity of this set, there exist $k \in \mathbb{N}$ and $\lambda_i \in [0, 1]$, $i = 0, 1, \ldots, k$, satisfying (2.2) such that $x_{\lambda_i} \in \mathcal{L}_\alpha(f + \xi)$, i.e.,

$$f(x_{\lambda_i}) + \xi(x_{\lambda_i}) \le \alpha = (1 - \lambda_i)(f(x_0) + \xi(x_0)) + \lambda_i(f(x_1) + \xi(x_1))$$

for $0 \le i \le k$. Due to the linearity of ξ again, this implies

$$f(x_{\lambda_i}) \le (1 - \lambda_i) f(x_0) + \lambda_i f(x_1) \text{ for } 0 \le i \le k.$$

Hence, f is outer γ-convex. ∎

If f is allowed to attain the value $-\infty$, then the necessity in the last proposition remains true, but the sufficiency does not. For instance, the function

$$f(x) := \begin{cases} 1/x & \text{if } x \in [-2, 0[\\ 0 & \text{if } x \in [0, 1[\\ -\infty & \text{if } x = 1 \end{cases}$$

is not outer γ-convex on $D = [-2, 1]$ for $\gamma = 2$ (to see this, choose $x_0 = -2$ and $x_1 = 1$) although the level set $\mathcal{L}_\alpha(f + \xi)$ is outer γ-convex for every continuous linear functional ξ on $\mathbb{R}$ and for every real number α (even for $\alpha = -\infty$).

Recall that $f : D \to \mathbb{R}$ is said to be *γ-convex like* if, for all x_0 and x_1 in D satisfying $\|x_0 - x_1\| > \gamma$, there exists $\lambda \in]0, 1[$ such that

$$f(x_\lambda) \le (1 - \lambda) f(x_0) + \lambda f(x_1).$$

[8]. Obviously, each lower level set of a γ-convex like function is γ-convex like, and each outer γ-convex function is γ-convex like but not vice versa. Nevertheless, the equivalence holds for lower semi-continuous functions.

Proposition 3.3. *Let $f : D \to \mathbb{R}$ be lower semi-continuous. Then f is outer γ-convex if and only if it is γ-convex like.*

Proof. Assume that f is γ-convex like. Then, for all $\xi \in X^*$, $f + \xi$ is γ-convex like and lower semi-continuous. Therefore, the level set $\mathcal{L}_\alpha(f + \xi)$ is always γ-convex like and relatively closed in D. Due to Proposition 2.5, $\mathcal{L}_\alpha(f + \xi)$ is outer γ-convex for all $\xi \in X^*$ and $\alpha \in \mathbb{R}$. Therefore, Proposition 3.2 implies that f is outer γ-convex. ■

Note that if $f : D \to \mathbb{R} \cup \{-\infty\}$, then the assertion of Proposition 3.3 is true, too. But in this case, one cannot use Proposition 3.2 to prove it.

Let us now consider the so-called *lower semi-continuous hull* of the function f defined by

$$\operatorname{lsc} f(x) := \liminf_{y \to x} f(x)$$

(where $y \in D$ may be equal to x). It is well known that this function is the greatest lower semi-continuous function on D majorized by f (compare with [9]).

Proposition 3.4. *If $f : D \to \mathbb{R} \cup \{-\infty\}$ is outer γ-convex, then $\operatorname{lsc} f$ is outer γ-convex, too.*

Proof. Let $x_0^*, x_1^* \in D$ with $\|x_0^* - x_1^*\| > \gamma$. By definition, there exist two sequences (x_0^n) and (x_1^n) in D satisfying

$$\lim_{n\to\infty} x_0^n = x_0^*, \quad \lim_{n\to\infty} f(x_0^n) = \operatorname{lsc} f(x_0^*),$$
$$\lim_{n\to\infty} x_1^n = x_1^*, \quad \lim_{n\to\infty} f(x_1^n) = \operatorname{lsc} f(x_1^*),$$
$$\|x_0^n - x_1^n\| \le \|x_0^* - x_1^*\| + \gamma, \ n = 1, 2, \ldots.$$

Since f is outer γ-convex, due to Proposition 2.2, there exist $\lambda_i^n \in [0, 1]$, $i = 0, 1, \ldots, k^* := 2\,\mathrm{rd}(\|x_0^* - x_1^*\|/\gamma) + 1$, satisfying

$$\lambda_0^n = 0, \quad \lambda_{k^*}^n = 1, \quad 0 \le \lambda_{i+1}^n - \lambda_i^n \le \frac{\gamma}{\|x_0^n - x_1^n\|} \ \text{for } i = 0, 1, \ldots, k^* - 1,$$

and

$$f(x^n_{\lambda^n_i}) \le (1-\lambda^n_i) f(x^n_0) + \lambda^n_i f(x^n_1) \text{ for } i = 0, 1, \ldots, k^*.$$

Similarly as in the proof of Proposition 2.4, we can assume, without loss of generality, that

$$\lim_{n\to\infty} \lambda^n_i = \lambda^*_i, \ i = 1, 2, \ldots, k^* - 1, \quad \lambda^*_0 = 0, \ \lambda^*_{k^*} = 1,$$

$$0 \le \lambda^*_{i+1} - \lambda^*_i \le \frac{\gamma}{\|x^*_0 - x^*_1\|} \text{ for } i = 0, 1, \ldots, k^* - 1.$$

Therefore, $\lim_{n\to\infty} x^n_{\lambda^n_i} = x^*_{\lambda^*_i}$ and

$$\operatorname{lsc} f(x^*_{\lambda^*_i}) \le \lim_{n\to\infty} f(x^n_{\lambda^n_i}) \le (1-\lambda^*_i)\operatorname{lsc} f(x^*_0) + \lambda^*_i \operatorname{lsc} f(x^*_1)$$

for $i = 0, 1, \ldots, k^*$. Hence, $\operatorname{lsc} f$ is outer γ-convex. ■

4. Optimization Properties

In the first part of this section, let γ be a non-negative real number. $x^* \in D$ is said to be a *γ-minimizer* or a *γ-infimizer* of the function $f : D \subseteq X \to \mathbb{R} \cup \{-\infty\}$ if there exists an $\varepsilon > 0$ such that $f(x^*) \le f(x)$ or $\liminf_{y\to x^*} f(y) \le f(x)$ for all $x \in D \cap B(x^*, \gamma+\varepsilon)$ ($B(x^*, r) := \{x \in X : \|x - x^*\| < r\}$), respectively. In particular, if $\gamma = 0$, then a γ-minimizer is a *local minimizer* and a γ-infimizer is a *local infimizer*. If $f(x^*) \le f(x)$ or $\liminf_{y\to x^*} f(y) \le f(x)$ for all $x \in D$, then x^* is a *global minimizer* or a *global infimizer*, respectively.

Proposition 4.1. *x^* is a γ-infimizer (or global infimizer) of f if and only if it is a γ-minimizer (or global minimizer, respectively) of $\operatorname{lsc} f$.*

Proof. By definition, x^* is a γ-infimizer of f if and only if if there exists an $\varepsilon > 0$ such that

$$\operatorname{lsc} f(x^*) = \liminf_{y\to x^*} f(y) \le f(x) \text{ for all } x \in D \cap B(x^*, \gamma+\varepsilon),$$

which holds if and only if

$$\operatorname{lsc} f(x^*) \le \liminf_{y\to x} f(y) = \operatorname{lsc} f(x) \text{ for all } x \in D \cap B(x^*, \gamma+\varepsilon),$$

i.e., if and only if x^* is a γ-minimizer of $\operatorname{lsc} f$. The rest is similar. ■

We will use the last assertion to prove the following main properties of outer γ-convex functions:

(M_γ) each γ-minimizer is a global minimizer,

(I_γ) each γ-infimizer is a global infimizer.

In the following, let $\gamma > 0$.

Proposition 4.2. *An outer γ-convex function possesses the properties (M_γ) and (I_γ), and it is absolutely stable with respect to them.*

Proof. Let $x_0 \in D \subseteq X$ and $f : D \longrightarrow \mathbb{R}$ be outer γ-convex. By definition, for all $x_1 \in D$, there is $\lambda_1 \in]0, \gamma/\|x_0 - x_1\|]$ such that

$$f(x_{\lambda_1}) \leq (1 - \lambda_1) f(x_0) + \lambda_1 f(x_1),$$

which implies $\|x_0 - x_{\lambda_1}\| \leq \gamma$ and

$$f(x_1) - f(x_0) \geq (f(x_{\lambda_1}) - f(x_0))/\lambda_1.$$

Therefore, if x_0 is a γ-minimizer of f, then $f(x_{\lambda_1}) \geq f(x_0)$, which yields that $f(x_1) \geq f(x_0)$ for all $x_1 \in D$, i.e., x_0 is a global minimizer of f.

Let x_0 now be a γ-infimizer of f. Due to Propositions 3.4 and 4.1, x_0 is a γ-minimizer of the outer γ-convex function $\operatorname{lsc} f$. By the assertion just proved, x_0 is a global minimizer of $\operatorname{lsc} f$. Consequently, Proposition 4.1 implies that x_0 is a global infimizer of f.

The absolute stability of an outer γ-convex function with respect to (M_γ) and (I_γ) follows then from Proposition 3.1. ■

Using Proposition 4.2, we can prove a stronger assertion, where the outer γ-convexity of f is not required.

Proposition 4.3. *If $\operatorname{lsc} f$ is outer γ-convex, then f possesses the properties (M_γ) and (I_γ), and f is absolutely stable with respect to them.*

Proof. If x^* is a γ-infimizer of f, then it is a γ-minimizer of the outer γ-convex function $\operatorname{lsc} f$ (Proposition 4.1), which implies that it is a global minimizer of $\operatorname{lsc} f$ (Proposition 4.2). Hence, by Proposition 4.1, x^* is a global infimizer of f, i.e., f possesses the property (I_γ).

If x^* is a γ-minimizer of f, then it is a γ-minimizer of $\operatorname{lsc} f$, too. Moreover, this fact also yields $f(x^*) = \operatorname{lsc} f(x^*)$. Since $\operatorname{lsc} f$ is outer γ-convex, Proposition 4.2 implies that x^* is a global minimizer of $\operatorname{lsc} f$. Therefore,

$$f(x^*) = \operatorname{lsc} f(x^*) \leq \operatorname{lsc} f(x) \leq f(x) \quad \text{for all } x \in D.$$

Hence, f possesses the property (M_γ).

For all continuous linear functional $\xi \in X^*$, we have $\operatorname{lsc}(f + \xi) = \operatorname{lsc} f + \xi$, which is outer γ-convex (due to Proposition 3.1). Following, $f + \xi$ possesses (M_γ) and (I_γ), i.e., f is absolutely stable with respect to them. ■

Propositions 4.2 and 4.3 ensure that any function $f : D \subseteq X \rightarrow \mathbb{R}$ is absolutely stable with respect to (M_γ) and (I_γ) if f or $\operatorname{lsc} f$ is outer γ-convex. This type of functions can be considered as the most general one having this property, because under some additional assumptions, this absolute stability is sufficient for the outer γ-convexity of f or $\operatorname{lsc} f$, as the following assertions show.

Proposition 4.4. *Let $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be lower semi-continuous. Then f is outer γ-convex if and only if $f + \xi$ possesses the property (M_γ) for every linear functional ξ on $\mathbb{R}$.*

Proof. By Theorem 3.1 in [8], a lower semi-continuous function $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ is γ-convex like if and only if, for every linear functional ξ on $\mathbb{R}$, each γ-minimizer of $f + \xi$ is a global minimizer. (The notion of γ-minimizer in [8] is a bit different from the one here, but the assertion remains true with the same proof.) Therefore, Proposition 3.3 implies the assertion. ■

Proposition 4.5. *Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be bounded from below. Then $\operatorname{lsc} f$ is outer γ-convex if and only if $f + \xi$ possesses the property (I_γ) for every linear functional ξ on $\mathbb{R}$.*

Proof. Since f is bounded from below, the function $\operatorname{lsc} f$ only attains its values in $\mathbb{R}$. Therefore, we can apply Proposition 4.4 for $\operatorname{lsc} f$, which yields that $\operatorname{lsc} f$ is outer γ-convex if and only if $\operatorname{lsc} f + \xi$ possesses the property (M_γ) for every linear functional ξ on $\mathbb{R}$. Since $\operatorname{lsc} f + \xi = \operatorname{lsc}(f + \xi)$, Proposition 4.1 yields that $\operatorname{lsc} f + \xi$ fulfills (M_γ) if and only if $f + \xi$ satisfies (I_γ). This completes our proof. ■

To conclude this section, let us consider the case $\gamma = 0$, i.e., local minimizers and local infimizers instead of γ-minimizers or γ-infimizers.

Proposition 4.6. *Let $f : D \subseteq X \to \mathbb{R}$. If f or $\operatorname{lsc} f$ is convex then, for all continuous linear functional $\xi \in X^*$, $f + \xi$ possesses the following properties:*

$$(M) \quad \text{each local minimizer is a global minimizer}$$

and

$$(I) \quad \text{each local infimizer is a global infimizer.}$$

Proof. It suffices to show that the convexity of $\operatorname{lsc} f$ implies the properties (M) and (I) of f. The rest follows from the fact that, if f is convex, then so are $\operatorname{lsc} f$ and $f + \xi$.

Assume now that $\operatorname{lsc} f$ is convex and $x^* \in D$ is a local minimizer or infimizer of f, i.e., there exists $\varepsilon > 0$ such that $f(x^*) \le f(x)$ or $\liminf_{y \to x^*} f(y) \le f(x)$ for all $x \in D \cap B(x^*, \varepsilon)$. Since $\operatorname{lsc} f$ is convex, it must be outer γ-convex with $\gamma := \varepsilon/2$. For this γ, x^* is a γ-minimizer or γ-infimizer of f, respectively. Therefore, Proposition 4.3 yields that x^* is a global minimizer or global infimizer of f, respectively. ■

Similar to Propositions 4.4 and 4.5, under some suitable assumptions, the absolute stability of f with respect to (M) or (I) becomes a sufficient condition for the convexity of f or $\operatorname{lsc} f$, respectively. In fact, Theorem 3.2 in [8] says that a lower semi-continuous function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is convex if and only if $f + \xi$ satisfies (M) for every linear function ξ on $\mathbb{R}$. Applying this and Proposition 4.1 leads by the same way as in the proof of Proposition 4.5 to the following:

Proposition 4.7. *Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be bounded from below. Then $\operatorname{lsc} f$ is convex if and only if, for every linear functional ξ on $\mathbb{R}$, $f + \xi$ possesses the property (I).*

Note that the assumption of boundedness from below in Propositions 4.5 and 4.7 is really needed. For instance, let

$$f(x) := \begin{cases} 1/(|x| - 1) & \text{if } x \in]-1, 1[\\ 0 & \text{if } |x| = 1, \end{cases}$$

$D = [-1, 1]$, and $\gamma = 1$. Then $f + \xi$ possesses the properties (I_γ) and (I) for every linear functional ξ on $\mathbb{R}$, but $\operatorname{lsc} f$ is neither convex nor outer γ-convex.

5 Concluding Remarks

In [4–8], we used the following weaker condition

$$f(x^*) \leq f(x) \quad \text{for all } x \in D \text{ satisfying } \|x - x^*\| \leq \gamma \tag{5.1}$$

to define γ-minimizers of f and also get the property (M_γ) for some kinds of roughly convex functions. If we do so in this paper, the assertion "x^* is a γ-infimizer of f if it is a γ-minimizer of $\operatorname{lsc} f$" in Proposition 4.1 does not hold anymore. For instance, let

$$f(x) = \begin{cases} 0 & \text{if } |x| \leq 1 \\ 1 - x^2 & \text{if } |x| > 1 \end{cases}$$

and $\gamma = 1$. Then $\operatorname{lsc} f = f$ and $x^* = 0$ is a γ-minimizer in the sense of (5.1) but not a γ-infimizer. Nevertheless, all other assertions in Sec. 4 remain true without changing their proof.

The quantity $\varepsilon > 0$ in the definition of γ-minimizer is really needed in order for the outer γ-convex functions to possess (I_γ). In fact, the function $f(x) = [x]$ (where $[x] := \max\{z \in \mathbb{Z} : z \leq x\}$) is outer γ-convex for $\gamma = 1$, and each $x^* \in \mathbb{Z}$ satisfies $x^* - 1 = \liminf_{y \to x^*} f(y) \leq f(x)$ for all $x \in [x^* - 1, x^* + 1]$, but x^* cannot be a global infimizer of f.

As pointed out in Sec. 4 of [8], we can define another kind of roughly convex functions as follows: For all $x_0, x_1 \in D$, there exist $\lambda_i \in [0, 1]$ such that

$$\begin{aligned} &\|(1 - \lambda_i)x_0 + \lambda_i x_1 - x_i\| \leq \gamma \quad \text{and} \\ &f((1 - \lambda_i)x_0 + \lambda_i x_1) \leq (1 - \lambda_i) f(x_0) + \lambda_i f(x_1) \end{aligned}$$

for $i = 0, 1$. Such a kind of roughly convex functions is weaker than outer γ-convex functions; nevertheless they also possess (M). But in general, these functions cannot satisfy (I) as the following example shows. The function

$$f(x) := \begin{cases} x & \text{if } x \in]-\infty, -1[\cup]1, +\infty[\\ 1 + 2|x| & \text{if } |x| \leq 1 \end{cases}$$

fulfills the above condition for $\gamma := 1.5$, and $x^* = 1$ is a γ-infimizer of f, but it cannot be a global infimizer.

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