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# On Stability of Systems of Linear Differential Equations on $\ell_\infty$-Space

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Stability problem of countable systems of differential equations has received much attention recently. Many interesting results were obtained in [1, 2]. In this paper we study stability of systems of linear differential equations on the space of bounded sequences of real numbers:

$$\ell_\infty = \{x = (x_1, x_2, \dots) : \|x\| = \sup_{i \in N} |x_i| < \infty\}.$$

**1.** Consider the system of linear differential equations on the $\ell_\infty$- space:

$$\frac{dx}{dt} = A(t)x, \tag{1}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}; \quad A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots \\ a_{21}(t) & a_{22}(t) & \dots \\ \dots & \dots & \dots \end{bmatrix}.$$

For the system (1) we assume the following conditions:

(i) the functions $a_{ij} : [a, +\infty) \to \mathbb{R}$ are continuous ($i, j \in \mathbb{N}$);
(ii) there exists a continuous positive function $\alpha(t)$ on $[a, +\infty)$ such that

$$\sum_{j=1}^{\infty} |a_{ij}(t)| \le \alpha(t) \quad (i \in \mathbb{N}, \ t \in [a, \infty)).$$

It is well-known that, under the above conditions, system (1) has a unique solution

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \end{bmatrix}$$

in which $x_1(t), x_2(t), \dots$ are equi-continuous on $[a, +\infty)$ (see [1, p. 25]).

In [2, p. 116] it was shown that system (1) has the fundamental matrix

$$X(t) = [x_{ij}(t)]_{i,j=1}^{\infty}$$

with

$$X(t_0) = \mathrm{Id}_E \quad \text{(the identity matrix).}$$

In fact, the set of columns of the fundamental matrix $X(t)$:

$$X_j(t) = \begin{bmatrix} x_{1j}(t) \\ x_{2j}(t) \\ \vdots \end{bmatrix}$$

$(j = 1, 2, \dots)$ is the fundamental system of solutions of (1).

Denote by $K(t, \tau)$ the resolvent operator (Cauchy operator) of (1). Clearly, $K(t, \tau) = X(t)X^{-1}(\tau)$.

Now, the solution $x(t) = x(t; t_0, x_0)$ of (1) with the initial condition $x(t_0) = x_0$ can be written as follows:

$$x(t) = K(t, t_0)x_0 .$$

For each $t \in [a, \infty)$, the matrix $X(t)$ defines the linear operator:

$$X(t)\colon \ell_\infty \to \ell_\infty$$
$$x_0 \mapsto x(t) = X(t)x_0 .$$

It is easy to see that

$$\|X(t)\| = \sup_{i \in N} \sum_{j=1}^{\infty} |x_{ij}(t)| ,$$

where $\|X(t)\|$ denotes the norm of the operator $X(t)$, i.e.,

$$\|X(t)\| = \sup_{\|x\|=1} \|X(t)x\| .$$

Moreover, the trivial solution $x = 0$ of (1) is (Lyapunov) stable if and only if the resolvent operator $K(t, t_0)$ is bounded, i.e., for each $t_0 \in [a, \infty)$, there exists $M = M(t_0)$ such that

$$\|K(t, t_0)\| \le M , t \ge t_0 .$$

**Definition 1.**

(a) *A continuous linear operator*

$$L(t)\colon \ell_\infty \to \ell_\infty \qquad (t \geq t_0)$$
$$y \mapsto L(t)y$$

*is said to be a generalized Lyapunov transformation if it is invertible, $L(t)$, $L^{-1}(t)$ are differentiable and $\chi[L(t)] = \chi[L^{-1}(t)] = 0$, where*

$$\chi[L(t)] := \overline{\lim_{t\to\infty}} \frac{1}{t} \ln \|L(t)\|$$

(b) A system of linear differential equations

$$\frac{dx}{dt} = A(t)x \tag{2}$$

is said to be regular if there is a generalized Lyapunov transformation

$$L(t) : x \mapsto L(t)x$$

transforming (2) to a system of linear differential equations with a constant matrix $B$

$$\frac{dy}{dt} = By\,. \tag{3}$$

**Theorem 1.** *Suppose (2) is regular and all characteristic exponents (see [3]) of its solutions are not greater than certain number $\alpha \in \mathbb{R}$, then for any $\varepsilon > 0$ and $t_0 \geq 0$, there exists a positive number $C$ such that*

$$\|K(t,\tau)\| \leq Ce^{(\alpha+\varepsilon)(t-\tau)}e^{\varepsilon\tau}$$

*for all $t \geq \tau \geq t_0$.*

**Theorem 2.** *Suppose (2) is regular and all characteristic exponents of its solutions are less than or equal to $-\alpha < 0$. Let $F(t,x)$ be a function belonging to $C_{tx}(R \times \ell_\infty)$ and satisfying the condition*

$$\|F(t,x)\| \leq \psi(t)\|x\|\,,$$

*where $\psi(t)$ is a function with $\chi[\psi(t)] < 0$.*

*Then the trivial solution $y = 0$ of the following system:*

$$\frac{dy}{dt} = A(t)y + F(t,y)$$

*is stable.*

2. In this section we shall study $J$-stability of systems of linear differential equations on $\ell_\infty$. Vu Tuan and Dang Dinh Chau [4] have introduced $J$-stability, which is recalled below.

For each $m \in N$, we set

$$\ell_\infty^m = \{x = (x_1, x_2, \dots, x_m, 0, \dots) \in \ell_\infty\}.$$

It is easy to see that $\ell_\infty^m$ is a closed subspace of $\ell_\infty$ with the norm induced from the norm on $\ell_\infty$.

Consider the linear operator

$$\begin{aligned} P_m \,:\, \ell_\infty &\to \ell_\infty \\ (x_1, x_2, \dots) &\mapsto (x_1, \dots, x_m, 0, \dots). \end{aligned}$$

Clearly, $P_m$ is a projection from $\ell_\infty$ onto $\ell_\infty^m$, $P_m(\ell_\infty) = \ell_\infty^m$.

For each fixed $t$, we may consider the fundamental matrix $X(t)$ of system (2) as the linear operator:

$$\begin{aligned} X(t) :\, \ell_\infty^m &\to \ell_\infty \\ P_m x &\mapsto X(t) P_m x\,. \end{aligned}$$

It is not hard to see that

$$\|X(t)\|_m := \sup_{\|P_m x\|=1} \|X(t)P_m x\| = \sup_{i\in\mathbf{N}} \sum_{j=1}^{m} |x_{ij}(t)|.$$

Let $J = \{n_1, n_2, ..., n_j ...\}$ be a strictly increasing subsequence of the sequence of positive integer numbers $\mathbf{N}$.

**Definition 2.** [4]

(a) *The trivial $x = 0$ of system (2) is called J-stable if, for any $\varepsilon > 0$, $t_0 \geq 0$ and any $m \in J$, there exists $\delta > 0$ such that*

$$\|x(t; t_0, P_m x_0)\| < \varepsilon$$

*for all $t \geq t_0$ and $x_0 \in \ell_\infty : \|P_m x_0\| < \delta$ (where $x(t; t_0, P_m x_0)$ is the solution of (2) with initial condition $x(t_0; t_0, P_m x_0) = P_m x_0$).*

(b) *The trivial solution $x = 0$ of system (2) is called J-uniformly stable if, for each $\varepsilon > 0$ and each $t_0 \geq 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that*

$$\|x(t; t_0, P_m x_0)\| < \varepsilon$$

*for all $t \geq t_0, m \in J$ and $x_0 \in \ell_\infty : \|P_m x_0\| < \delta$.*

**Theorem 3.** *The trivial solution $x = 0$ of (2) is Lyapunov stable if and only if it is J-uniformly stable.*

## References

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