Vietnam Journal of Mathematics **27**:4 (1999) 373–377

Vietnam Journal of MATHEMATICS

Short Communication

Stability of Nonlinear Discrete Time-Varying Retarded Systems

Nguyen Sinh Bay and Vu Ngoc Phat

Institute of Mathematics, P.O. Box 631, Bo Ho, Hanoi, Vietnam

Received July 14, 1998
Revised July 25, 1999

In recent years, the stability problem of discrete time-varying retarded systems has been widely studied (see, e.g., [1–5]). For nonlinear discrete systems, the stability problem was mainly treated by using the Lyapunov function, which is, in many cases, difficult to find. In this paper, we will establish some verifiable sufficient conditions for asymptotic stability of discrete time-varying systems, without using the Lyapunov functions approach. Our result extends the result of [6] to more general nonlinear equations and to the systems with delays.

1. Stability of General Nonlinear Systems

Consider a nonlinear discrete time system of the form

$$x(k+1) = f(k, x(k)), \tag{1}$$

where $k \in Z^+$, $x \in R^n$, $f : Z^+ \times R^n \to R^n$ is a given nonlinear function satisfying $f(k, 0) = 0$ for all $k \in Z^+$.

The discrete system (1) with the initial condition $x(0) = x_0$ always has the solution $x(k)$, which is defined as

$$x(k) = f(k-1,\ f(k-2, ..., f(0, x_0)...).$$

Moreover, the system (1) has the zero solution $x(k) = 0$, $\forall k \in Z^+$.

Let the system (1) satisfy the following condition: There exist a positive integer m and real numbers $\alpha_i \geq 0;\ p_i \geq 0,\ i = 1, 2, ..., m;\ p_1 < p_2 < \cdots < p_m$ such that

$$\|f(k, x)\| \leq \sum_{i=1}^{m} \alpha_i \|x\|^{p_i} \tag{2}$$

for all $x \in R^n$ and $k \in Z^+$. As will be shown in this paper, inequalities $p_1 \geq 1$; $p_1 > 1$ or $p_1 > 0$ and the relation between p_1 and numbers α_i have an important meaning for the stability of the system. In fact, the following results hold:

Theorem 1. [6] *System (1) satisfying the assumption (2) is asymptotically stable if* $p_1 \geq 1$ *and* $\alpha_1 < 1$.

It is not difficult to show that the above theorem still holds for any arbitrary number $\alpha_1 > 0$, but $p_1 > 1$.

For the case, where $p_1 > 0$, the following theorem gives another criterion for asymptotic stability of system (1) with assumption (2), where the numbers α_i depend on k, i.e.,

$$\|f(k, x)\| \leq \sum_{i=1}^{m} \alpha_i(k)\|x\|^{p_i}. \tag{3}$$

Theorem 2. *System (1) satisfying the assumption (3) where* $m \in Z^+$; $p_i > 0$; $p_1 < p_2 < \cdots < p_m$, $\alpha_i(k) : Z^+ \to R^+$; $i = 1, 2, ..., m$ *is asymptotically stable if* $\lim_{k\to\infty} \alpha_i(k) = 0$ *for all* $i = 1, 2, ..., m$.

The proof is similar to that of Theorem 1, by successive estimating $x(k)$, $k = 0, 1, 2, \ldots$.

2. Stability of Nonlinear Systems with Delays

Consider the nonlinear discrete time-varying system with delays of the form

$$\begin{aligned} x(k+1) &= f(k, x(k), x(k-r)), \\ 0 &= f(k, 0, 0), \end{aligned} \tag{4}$$

where $x \in R^+$, $r \in Z^+$, $k \in Z^+$. We will consider the system with the following initial condition:

$$x(k) = x_k^0, \text{ for } k = -r;\ -r+1, ..., -1, 0. \tag{4a}$$

The zero solution of system (4) is said to be stable if, for every $\varepsilon > 0$, there exist a number $\delta > 0$ such that the initial condition (4a) with $\|x_r^0\| < \delta$, $\|x_{r-1}^0\| < \delta$, ..., $\|x_1^0\| < \delta$, $\|x_0^0\| < \delta$ implies $\|x(k)\| < \varepsilon$, for all $k \in Z^+$. If, moreover, $\|x(k)\| \to 0$ as $k \to \infty$, then the zero solution is said to be asymptotically stable.

For system (4), the following assumption is made on the right-hand side function $f(\cdot)$:

$$\|f(k, x, y)\| \leq \sum_{i=1}^{m} \alpha_i \|x\|^{p_i} \|y\|^{q_i}, \tag{5}$$

where $m \in Z^+$, $\alpha_i > 0$, $p_i \geq 0$, $q_i \geq 0$; $i = 1, 2, ..., m$,

$$p_1 + q_1 < p_2 + q_2 < \cdots < p_m + q_m.$$

Theorem 3. *Assume condition (5) holds. System (4) is asymptotically stable if one of the following two conditions is satisfied:*

(a) $p_1 + q_1 > 1$ and $\alpha_1 > 0$;
(b) $p_1 + q_1 \geq 1$ and $0 < \alpha_1 < 1$.

Proof. (a) Let $\delta \in (0, 1)$ be chosen and $x(k) = x_k^0$ for all $k = -r, -r+1, ..., -1, 0$ such that $\|x(k)\| < \delta$, for all $k = -r, -r+1, ..., -1, 0$. From the condition $p_1 + q_1 > 1$, one can take a number $s > 1$ such that

$$p_1 + q_1 s^{-r} - s^r > 0.$$

This implies

$$p_1 + q_1 s^{-r} - s > 0 \quad \text{and} \quad p_1 + q_1 - s > 0.$$

Taking $\delta \in (0, 1)$ small enough such that

$$\sum_{i=1}^{m} \alpha_i \delta^{p_1 + q_1 s^{-r} - s} \leq 1,$$

we obtain

$$\sum_{i=1}^{m} \alpha_i \delta^{p_1 + q_1 - s} \leq 1.$$

By induction, we can prove that $\|x(k)\| \leq \delta^{s^k}$. Indeed, for $k = 1$, we have

$$\begin{aligned}
\|x(1)\| &\leq \sum_{i=1}^{m} \alpha_i \|x(0)\|^{p_i} \|x(-r)\|^{q_i} \\
&\leq \sum_{i=1}^{m} \alpha_i \delta^{p_i + q_i} \\
&\leq \sum_{i=1}^{m} \alpha_i \delta^{p_1 + q_1 - s} \delta^{s} < \delta^{s^1}.
\end{aligned}$$

Let the above estimation be true for $k = N$. Then, for $k = N + 1$, we have

$$\begin{aligned}
\|x(N+1)\| &\leq \sum_{i=1}^{m} \alpha_i \|x(N)\|^{p_i} \|x(N-r)\|^{q_i} \\
&\leq \sum_{i=1}^{m} \alpha_i \delta^{s^N p_i} \delta^{s^{N-r} q_i} \\
&\leq \sum_{i=1}^{m} \alpha_i \delta^{s^N (p_1 + q_1 s^{-r} - s)} \delta^{s^{N+1}} \\
&\leq \sum_{i=1}^{m} \alpha_i \delta^{p_1 + q_1 s^{-r} - s} \delta^{s^{N+1}} \leq \delta^{s^{N+1}}.
\end{aligned}$$

From estimation $\|x(k)\| \leq \delta^{s^k}$, for all $k \in Z^+$ where $s > 1$, $\delta \in (0, 1)$, we have $\|x(k)\| \to 0$ as $k \to \infty$ which implies that system (4) is asymptotically stable.

Case (b) is proved similarly, by using the assumption $\alpha_1 < 1$.

In the case, where the numbers α_i depend on k, based on the result of Theorem 2, we can replace (5) by the following:

$$\|f(k, x, y)\| \leq \sum_{i=1}^{m} \alpha_i(k)\|x\|^{p_i}\|y\|^{q_i}. \tag{6}$$

Theorem 4. *The retarded system (4) satisfying (6), where $p_i > 0$, $q_i > 0$, $i = 1, 2, ..., m$, is asymptotically stable if*

$$\lim_{k\to\infty} \alpha_i(k) = 0 \quad \text{for all} \quad i = 1, 2, \ldots, m.$$

Theorems 3 and 4 can be extended to the case of more general retarded systems of the form

$$\begin{aligned} x(k+1) &= f(k, x(k),\ x(k-1), \ldots, x(k-r)), \\ 0 &= f(k,\ 0,\ 0, \ldots, 0). \end{aligned} \tag{7}$$

Indeed, we assume that (7) satisfies the following condition:

$$\|f(k, x_1, x_2, \ldots, x_{r+1})\| \leq \sum_{i=1}^{m} \alpha_i \prod_{j=1}^{r+1} \|x_j\|^{p_{ij}}. \tag{8}$$

Then, by the same argument as for Theorem 3, we can prove the following result.

Theorem 5. *If system (7) satisfies condition (8), where $\alpha_i > 0$, $p_{ij} > 0$, $i = 1, 2, ..., m$, $j = 1, 2, ..., r+1$ and*

$$p_{11} + p_{12} + \cdots + p_{1r+1} < p_{21} + p_{22} + \cdots + p_{2r+1} < \cdots < p_{m1} + p_{m2} + \cdots + p_{mr+1},$$

then system (7) is asymptotically stable if one of the two following conditions holds:

(a) $p_{11} + p_{12} + \cdots + p_{1r+1} > 1$ *and* $\alpha_1 > 0$;
(b) $p_{11} + p_{12} + \cdots + p_{1r+1} \geq 1$ *and* $\alpha_1 < 1$.

As an illustration, we can show, by using the above theorems, that the following systems are asymptotically stable:

(1) $$x(k+1) = \begin{cases} 0, & \text{if } x(k) = 0 \\ \dfrac{\ln(k^2+1)x(k)}{(k+1)^{\alpha}\left[\max\limits_{1\leq i\leq n} \{x_i^2\}\right]^{1/3}} & \end{cases}$$

where $x \in R^n$; $\alpha > 0$.

$$(2) \qquad \begin{cases} x_1(k+1) = a_1 x_1(k) x_2(k-r) \\ x_2(k+1) = a_2 x_2(k) x_3(k-r) \\ x_3(k+1) = a_3 x_3(k) x_1(k-r) \end{cases}$$

where $x = (x_1, x_2, x_3) \in R^3$; $a_1, a_2, a_3 > 0$; $r \in Z^+$.

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