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A Remark on Non-Uniform Property of Linear Cocycles

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Abstract. We show that there are open sets of non-uniformly hyperbolic cocycles in the space of linear cocycles equipped with L^∞ topology.

1. Introduction

A discrete-time linear deterministic dynamical system is defined by a single matrix A, and its Lyapunov spectrum is nothing but the set of the real parts of the eigenvalues of A. The object of our interest in this paper are products of random matrices (linear cocycles). Thanks to the Multiplicative Ergodic Theorem of Oseledets [10], the Lyapunov spectrum of a cocycle is well defined (under some integrability conditions) and it is a generalization of the Lyapunov spectrum in the deterministic case.

The study of the Lyapunov spectrum of linear cocycles is one of central tasks of the theory of random dynamical systems. In various situations it is of great theoretical and practical importance to know when the Lyapunov spectrum is simple (see Arnold [1]). Another simpler question is whether a given cocycle is hyperbolic. Recall that a linear cocycle which satisfies the integrability conditions of the Multiplicative Ergodic Theorem of Oseledets is called *hyperbolic* if none of its Lyapunov exponents vanishes.

In the particular case of a product of independent and identically distributed random matrices, the Lyapunov spectrum is fairly well investigated and strong results are obtained (see [2, 5, 6]). We mention that in this case, the cocycles with simple Lyapunov spectrum form a residual set [2].

However, in the general case not much has been done. In the two-dimensional case, Knill [8] has proved that the cocycles with simple Lyapunov spectrum form a dense set in the space of all bounded cocycles equipped with the uniform topology. Recently, Fabbri [4] has investigated the problem of hyperbolicity of *two-dimensional continuous-time* cocycles generated by differential equations on tori and obtained the density results with respect to C^r-topology, $0 \le r < \infty$. Note that the C^0-result of Fabbri is a continuous-time version of the abovementioned result of Knill [8].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a non-atomic probability space and θ an ergodic automorphism of $(\Omega, \mathcal{F}, \mathbb{P})$ preserving the probability measure $\mathbb{P}$. A measurable mapping A from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the topological space $Gl(d, \mathbb{R})$ (for short: $Gl(d)$) of linear non-singular operators of $\mathbb{R}^d$ equipped with its Borel σ-algebra is called a *random linear map*. A generates a *linear cocycle* over the dynamical system θ via

$$\Phi_A(n,\omega) := \begin{cases} A(\theta^{n-1}\omega) \circ \cdots \circ A(\omega), & n > 0, \\ \mathrm{id}, & n = 0, \\ A^{-1}(\theta^{n}\omega) \circ \cdots \circ A^{-1}(\theta^{-1}\omega), & n < 0. \end{cases}$$

Conversely, if we are given a linear cocycle over θ, then its time-one map is a linear random map. Therefore, the correspondence between A and Φ_A is one-to-one and we are free to choose one from them to work with. We also speak of a linear cocycle A, meaning the cocycle Φ_A generated by A.

We shall look at linear cocycles as linear operators of $\mathbb{R}^d$ and identify them with their matrix representations in the standard Euclidean basis of $\mathbb{R}^d$. We denote by $Gl(d)$ the group of non-singular d-dimensional matrices.

Since we deal with discrete-time cocycles we can always neglect sets of null measure, and we shall identify the random mappings which coincide $\mathbb{P}$-almost surely.

Denote by $\mathcal{G}(d)$ the set of all $Gl(d)$-valued random maps. We define a metric ρ on $\mathcal{G}(d)$ such that $(\mathcal{G}(d), \rho)$ can be considered as a version of $L^\infty(\mathbb{P})$. For $A, B \in \mathcal{G}(d)$, set

$$\delta(A,B) := \operatorname*{ess\,sup}_{\omega\in\Omega} \|A(\omega) - B(\omega)\| + \operatorname*{ess\,sup}_{\omega\in\Omega} \|A^{-1}(\omega) - B^{-1}(\omega)\|$$

and

$$\rho(A,B) := \begin{cases} \delta(A,B)(1+\delta(A,B))^{-1} & \text{if } \delta(A,B) < \infty, \\ 1 & \text{if } \delta(A,B) = \infty. \end{cases}$$

It is known that $(\mathcal{G}(d), \rho)$ is a complete metric space (see [2]).

2. Non-Uniform Hyperbolicity - Exponential Dichotomy

Definition 2.1. *We say that the linear cocycle $\Phi_A(n,\omega)$ generated by a linear random map $A(\cdot)$ exhibits an exponential dichotomy if there exist positive numbers $K > 0$, $\alpha > 0$ and a family of projections P_ω of $\mathbb{R}^d$ depending measurably on $\omega \in \Omega$ such that*

(i) $\|\Phi_A(n,\omega)P_\omega\Phi_A^{-1}(m,\omega)\|_{\theta^m\omega,\theta^n\omega} \le K\exp(-\alpha(n-m))$ *for all* $n \ge m$, $\omega \in \Omega$,

(ii) $\|\Phi_A(n,\omega)(\mathrm{id} - P_\omega)\Phi_A^{-1}(m,\omega)\|_{\theta^m\omega,\theta^n\omega} \le K\exp(-\alpha(n-m))$ *for all* $n \le m$, $\omega \in \Omega$.

We would like to distinguish two features of exponential dichotomy which make it a useful tool in investigating cocycles. First, it is a *uniform property*, i.e., a cocycle with exponential dichotomy exhibits a *uniform hyperbolic structure*, their trajectories uniformly exponentially go to zero (forwards or backwards in time) with constants K, α independent of ω. Second, it is a robust property, i.e., all the cocycles close to a cocycle with exponential dichotomy will also have exponential dichotomy. The roughness property of exponential dichotomy was first proved by Coppel [3] and developed by Palmer [11] for deterministic dynamical systems; a random version of Coppel's theorem applied for cocycles was given by Gundlach [7] and Nguyen Dinh Cong [9].

We call a linear cocycle *uniformly hyperbolic* if it exhibits an exponential lichotomy. If a cocycle does not exhibit an exponential dichotomy but is hyper-olic, then we say that it is *non-uniformly hyperbolic*.

ı. An Open Set of Non-Uniformly Hyperbolic Linear Cocycles

n this section we construct an open set of non-uniformly hyperbolic linear co-ycles.

emma 3.1. *Assume that the probability space $(\Omega, \mathbb{P})$ is a non-atomic Lebesgue pace. Then there exists a measurable set $U \subset \Omega$ which can be represented in the rm*

$$U = \bigcup_{k=0}^{\infty} \bigcup_{j=0}^{k} \theta^j U_k ,$$

here the sets $\theta^j U_k$, $k = 0, 1, \ldots$, $j = 0, \ldots, k$, are pairwise disjoint and are all positive $\mathbb{P}$-measure.

roof. Since $(\Omega, \mathbb{P})$ is a non-atomic Lebesgue space the Rohlin–Halmos Lemma applicable to it, so that for any $\varepsilon > 0$ and any $n \in \mathbb{N}$, there is $V_{\varepsilon,n} \subset \Omega$ such at

i) $V_{\varepsilon,n}$ is measurable,

i) the sets $\theta^m V_{\varepsilon,n}$, $m = 0, \ldots, n-1$, are disjoint,

i) $\sum_{m=0}^{n-1} \mathbb{P}(\theta^m V_{\varepsilon,n}) > 1 - \varepsilon$.

ıt

$$V := \bigcup_{n=4}^{\infty} V_{1/8,n}.$$

is easily seen that V can be represented in the form

$$V = \bigcup_{k=0}^{\infty} \bigcup_{j=0}^{k} \theta^j \tilde{V}_k ,$$

ere the sets $\theta^j \tilde{V}_k$, $k = 0, 1, \ldots$, $j = 0, \ldots, k$, are pairwise disjoint, and in-tely many numbers of them, say $V_{k_1}, V_{k_2}, \ldots$, are of positive $\mathbb{P}$-measure. Setting $U_m := V_{k_m}$ and

$$U := \bigcup_{k=0}^{\infty} \bigcup_{j=0}^{k} \theta^j U_k ,$$

we see that the set U satisfies the conclusion of the lemma. ∎

Proposition 3.2. *Assume that the probability space $(\Omega, \mathbb{P})$ is a non-atomic Lebesgue space. Then there is an open set $Q \subset \mathcal{G}(1)$ such that any cocycle $A \in Q$ is non-uniformly hyperbolic.*

Proof. By Lemma 3.1, there is a measurable set $U \subset \Omega$ which can be represented in the form

$$U = \bigcup_{k=0}^{\infty} \bigcup_{j=0}^{k} \theta^j U_k ,$$

where the sets $\theta^j U_k$, $k = 0, 1, \ldots$, $j = 0, \ldots, k$, are pairwise disjoint and are all of positive $\mathbb{P}$-measure. Clearly we can choose U such that $\mathbb{P}(U) < 1/4$ because otherwise we can cut off the sets $\theta^j U_k$.

Construct a cocycle $A_0 \in \mathcal{G}(1)$ as follows:

$$A(\omega) := \begin{cases} 1/3 & \text{if } \omega \in U, \\ 3 & \text{if } \omega \in \Omega \setminus U. \end{cases}$$

Since A is bounded it satisfies the integrability condition of the Multiplicative Ergodic Theorem of Oseledets (see [10]). The Lyapunov exponent of A is determined in this one-dimensional case by the following formula:

$$\lambda_A = \int_{\Omega} \log A(\omega) d\mathbb{P}(\omega).$$

Because $\mathbb{P}(U) < 1/4$, by the choice of U, we have

$$\lambda_A > 3/4 - 1/4 = 1/2 > 0,$$

hence, A is hyperbolic. Define an open set Q in $\mathcal{G}(1)$ as follows:

$$Q := \{B \in \mathcal{G}(1) : \rho(B, A) < 1/13\}.$$

Now, let $B \in Q$ be arbitrary. Then

$$1/4 < B(\omega) < 5/12 \text{ if } \omega \in U,$$
$$3 - 1/12 < B(\omega) < 3 + 1/12 \text{ if } \omega \in \Omega \setminus U.$$

It is easily seen that B has positive Lyapunov exponent $\lambda_B > 0$, and hence is hyperbolic. Furthermore, since iterates of $B(\omega)$ decrease exponentially on U and U has infinitely long segments of orbits and $\lambda_B > 0$, B cannot exhibit an exponential dichotomy. This completes the proof of the proposition. ∎

Remark.

(i) One can easily adapt the arguments of the proof of Proposition 3.2 to the higher dimensional case and get an open set of non-uniformly hyperbolic cocycles in $\mathcal{G}(d)$ for any $d \in \mathbb{N}$.

(ii) It is natural that randomness of the ergodic dynamical system (Ω, θ) induces randomness of linear cocycles built over it. This is a reason for non-uniform property of linear cocycles shown by Proposition 3.2.

(iii) L.-S. Young [2] studied $Sl(2)$-valued cocycles and constructed an open set of non-uniformly hyperbolic cocycles. Her approach estimates Lyapnov exponents of $Sl(2)$-valued cocycles which is different from ours.

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