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Short Communication

Local Homology for Linearly Compact Modules

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1. Introduction

Let I be an ideal of a Noetherian commutative ring R and M an R-module. In [3] we introduced the concept of *local homology modules* $H_i^I(M)$ of M with respect to I, which is defined by $H_i^I(M) = \varprojlim_t \mathrm{Tor}_i^R(R/I^t; M)$ $(i \geq 0)$. Also in [3], we have shown some fundamental properties of local homology modules when M is Artinian. Since Artinian modules are linearly compact with discrete topology [11], there is a natural question: How to define a local homology theory for linearly compact modules? Note that the concept of linearly compact spaces was first introduced by Lefschetz [9] for vector spaces of infinite dimension and it was then generalized for modules by Zelinsky [16]. It was also studied by several authors: H. Leptin, I. G. Macdonald, C. U. Jensen, H. Zöschinger, et al.. The purpose of this note is to give basic results about local homology of linearly compact modules.

Throughout, the ring R is commutative, Noetherian, and has a topological structure.

2. Linearly Compact Modules

In this section we recall the concept of *linearly compact module* by the terminology of I. G. Macdonald [11] and some of its basic properties.

Let M be a topological R-module. A *nucleus* of M is a neighborhood of the zero element of M, and a *nuclear base* of M is a base for the nuclei of M. If N is a submodule of M which contains a nucleus, then N is open (and therefore closed) in M, and M/N is discrete. M is Hausdorff if and only if the intersection of all the nuclei of M is 0. M is said to be *linearly topologized* if M has a nuclear base $\mathcal{M}$ consisting of submodules.

A Hausdorff linearly topologized R-module M is *linearly compact* if M has the following property: If $\mathcal{F}$ is a family of closed cosets (i.e., cosets of closed submodules) in M which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection.

If M is an Artinian R-module, then M is linearly compact and discrete.

We first show that if M is linearly compact, then the functor $\mathrm{Tor}_i^R(-;M)$ transforms an inverse system of finitely generated modules into an inverse system of linearly compact modules with continuous homomorphisms.

Proposition 2.1. *Let $\{N_t\}$ be an inverse system of finitely generated R-modules and M a linearly compact R-module. Then $\{\mathrm{Tor}_i^R(N_t;M)\}_t$ $(i \geq 0)$ forms an inverse system of linearly compact R-modules and homomorphisms are continuous.*

The following proposition shows that $\varprojlim$ can commute to Tor for inverse systems of linearly compact modules.

Proposition 2.2. *If N is a finitely generated R-module and $\{M_t\}_t$ an inverse system of linearly compact R-modules with continuous homomorphisms, then for all $i \geq 0$, $\{\mathrm{Tor}_i^R(N;M_t)\}$ forms an inverse system of linearly compact modules with cotinuous homomorphisms. Moreover, we have an isomorphism*

$$\mathrm{Tor}_i^R(N;\varprojlim_t M_t) \cong \varprojlim_t \mathrm{Tor}_i^R(N;M_t).$$

3. Linearly Compact Local Homology Modules

We first recall the definition of local homology modules in [3, 3.1].

Definition 3.1. *Let I be an ideal of R and M an R-module. For all $i \geq 0$, the ith local homology module $H_i^I(M)$ of M with respect to I is defined by*

$$H_i^I(M) = \varprojlim_t \mathrm{Tor}_i^R(R/I^t;M).$$

Remarks.

(i) If M is a linearly compact $R-$module, so is $H_i^I(M)$.

(ii) If the ideal I is generated by r elements $x_1,\dots,x_r$ in R, then

$$H_i^I(M) \cong \varprojlim_t H_i(\underline{x}(t);M),$$

where $H_i(\underline{x}(t);M)$ is the ith Koszul homology module of M with respect to the system $\underline{x}(t) = (x_1^t,\dots,x_r^t)$.

Let $L_i^I(M)$ be the i-th derived module of the I-adic completion $\Lambda_I(M) = \varprojlim_t M/I^tM$ of M. The following theorem shows that our Definition 3.1 is coincidential with the definition of Greenlees and May [6, 2.4] when M is linearly compact.

Theorem 3.2. *If M is a linearly compact R-module, then, for all $i \geq 0$,*

$$H_i^I(M) \cong L_i^I(M).$$

The following result is an immediate consequence of Theorem 3.2.

Corollary 3.3. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of linearly compact modules. Then we have a long exact sequence of local homology modules

$$\cdots \longrightarrow H_i^I(M') \longrightarrow H_i^I(M) \longrightarrow H_i^I(M'') \longrightarrow$$

$$\cdots \longrightarrow H_0^I(M') \longrightarrow H_0^I(M) \longrightarrow H_0^I(M'') \longrightarrow 0.$$

An R-module M is called I-separated if $\bigcap_{t>0} I^t M = 0$. The following proposition says that the local homology module $H_i^I(M)$ is I-separated.

Proposition 3.4. *Let M be an R-module. Then, for all $i \geq 0$,*

$$\bigcap_{t>0} I^t H_i^I(M) = 0.$$

The following theorem gives us a characterization of I-separated modules.

Theorem 3.5. *Let M be a linearly compact R-module. The following statements are equivalent:*

(i) *M is I-separated, i.e., $\bigcap_{t>0} I^t M = 0$.*
(ii) *$\Lambda_I(M) \cong M$.*
(iii) *$H_0^I(M) \cong M$, $H_i^I(M) = 0$ for all $i > 0$.*

To state the next theorem, we recall notions of *co-associated* prime ideals and *magnitude* of a module. A prime ideal p is called *co-associated* to a non-zero module M if there is an Artinian homomorphic image L of M with $p = \mathrm{Ann}L$. We write $\mathrm{Coass}M$ for the set of co-associates primes (see [17]). The magnitude $\mathrm{mag}M$ of an R-module M is defined by $\mathrm{mag}M = \mathrm{Sup}\{\dim R/p \mid p \in \mathrm{Coass}M\}$ (see [15, 2.1]). Note that for an arbitrary linearly compact module, $\mathrm{mag}M \leq \dim R/\mathrm{Ann}M$ and there are some examples which show that $\mathrm{mag}M < \dim R/\mathrm{Ann}M$.

Theorem 3.6. *Let M be a linearly compact R-module with $\mathrm{mag}M = d$. Then, for all $i > d$,*

$$H_i^I(M) = 0.$$

4. Duality

In this section, (R, m) shall be a local Noetherian ring, m its maximal ideal and $k = R/m$ its residue field. Suppose now that the topology on R is the m-adic topology.

We first observe that $H_i^{m}(M)$ has a natural module structure over the m-adic completion $\widehat{R}$ of R for all $i \geq 0$. The first main result in this section is the Noetherian property of local homology modules. Note that a Hausdorff linearly topologized R-module M is called *semi-discrete* if every submodule of M is closed.

Theorem 4.1. *If M is a semi-discrete linearly compact R-module, then $H_i^{m}(M)$ is a Noetherian $\widehat{R}$-module for all $i \geq 0$.*

Let $D(M) = \mathrm{Hom}_R(M, E(R/m))$ be the Matlis dual of M. We have the following duality between local cohomology modules $H_I^i(M)$ and local homology modules $H_i^I(M)$.

Theorem 4.2. *Let M be an R-module. Then for all $i \geq 0$,*

$$H_i^I(D(M)) \cong D(H_I^i(M)).$$

When (R, m) is a complete local ring we have

Corollary 4.3. *Let (R, m) be a complete local ring and M a linearly compact semi-discrete R-module. Then for all $i \geq 0$,*

$$H_i^I(M) \cong D(H_I^i(D(M))).$$

In the case where R is a complete local ring, the class of linearly compact semi-discrete R-modules contains all Noetherian R-modules. Therefore, the following consequence is a generalization of a well-known result, which says that local modules $H_I^i(M)$ of a Noetherian R-module M are Artinian.

Corollary 4.4. *Let (R, m) be a complete local ring and M a linearly compact semi-discrete R-module. Then, for all $i \geq 0$, $H_{m}^i(M)$ is Artinian.*

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