Vietnam Journal of Mathematics **28**:1 (2000) 25-31

Vietnam Journal of MATHEMATICS

On Solvability in a Closed Form of a Class of Singular Integral Equations with Rotation and a Regular Part

Nguyen Tan Hoa

Department of Mathematics, Gia Lai Teachers' Training College
Gia Lai, Vietnam

Received October 16, 1998
Revised June 4, 1999

Abstract. In this paper we study the solvability in a closed form of singular integral equations of certain form. The method in this report is to reduce such equations to systems of singular integral equations of Cauchy type and then obtain all solutions in a closed form.

1. Introduction

Let Γ be a simple regular closed arc on the complex plane $\mathbf{C}$. It is known that the equation of the form

$$a(t)\varphi(t) + b(t)(S\varphi)(t) = f(t)$$

admits an effective solution (in a closed form), where S is a singular integral operator of Cauchy type in $H^\mu(\Gamma)$ $(0 < \mu < 1)$ (see [5]).

In [2], Ng. V. Mau considered the problem of solvability in closed form for singular integral equations of the form

$$\varphi(t) + \frac{1}{\pi i}\int_\Gamma \frac{\tau^{n-1-k}t^k}{\tau^n - t^n} M(\tau,t)\varphi(\tau)d\tau = f(t),$$

where n, k are non-negative integers, $0 \le k \le n-1$, and Γ is the unit circle on the complex plane.

Let $\Gamma = \{t \in \mathbf{C} : |t| = 1\}$, $D^+ = \{z \in \mathbf{C} : |z| < 1\}$, $D^- = \{z \in \mathbf{C} : |z| > 1\}$. Denote by X the space $H^\mu(\Gamma)$ $(0 < \mu < 1)$.

Consider a singular integral equation of the form

$$\varphi(t) + \frac{1}{\pi i}\int_\Gamma \frac{\tau^{n-1-k}\, t^k}{\tau^n + t^n} M(\tau,t)\varphi(\tau)d\tau = f(t)\,, \tag{1}$$

where $\varphi(t)$, $f(t) \in X$ and $M(\tau, t)$ is a function satisfying Hölder's condition in both variables $(\tau, t) \in \Gamma \times \Gamma$, $0 \le k \le n-1$, $1 \le n \in N$.

In this paper we study solvability in a closed form of singular integral equations of the form (1).

By algebraic method we reduce Eq. (1) to a system of singular integral equations of Cauchy type and then obtain all solutions in a closed form.

2. Preliminaries

Let

$$(S\varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - t} d\tau,$$

$$(\widetilde{S}_{n,k}\varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\tau^{n-1-k} t^k}{\tau^n + t^n} \varphi(\tau) d\tau, \quad 0 \le k \le n-1, \tag{2}$$

$$(W\varphi)(t) = \varphi(\varepsilon_1 t), \quad \varepsilon_1 = \exp\left(\frac{\pi i}{n}\right), \quad \varepsilon_j = \varepsilon_1^j \quad (j = 1, ..., 2n).$$

We have (see [5]) $S^2 = I$, $W^{2n} = I$, where I is the identity operator on X.

Denote

$$P = \frac{1}{2}(I + S), \quad Q = \frac{1}{2}(I - S),$$

$$P_j = \frac{1}{2n} \sum_{\nu=1}^{2n} \varepsilon_j^{2n-1-\nu} W^{1+\nu} \quad (j = 1, ..., 2n).$$

Then we have (see [5])

$$P^2 = P, \; Q^2 = Q, \; PQ = QP = 0, \; P_i P_j = \delta_{ij} P_j \; (i, j = 1, ..., 2n),$$

$$I = \sum_{j=1}^{2n} P_j, \quad W^k = \sum_{j=1}^{2n} \varepsilon_j^k P_j, \tag{3}$$

$$X = X^+ \oplus X^- = \bigoplus_{j=1}^{2n} X_j,$$

where $X^+ = PX$, $X^- = QX$, $X_j = P_j X$ $(j = 1, ..., 2n)$, δ_{ij} is the Kronecker symbol.

Lemma 1. *Let $\widetilde{S}_{n,k}$ be of the form (2). Then*

$$\widetilde{S}_{n,k} = SP_k - SP_{n+k} \; (k = 0, .., n-1),$$

where we admit $P_0 = P_{2n}$.

Proof. From the identity

$$\frac{\tau^{n-1-k}t^k}{\tau^n+t^n}=\frac{\tau^{2n-1-k}t^k}{\tau^{2n}-t^{2n}}-\frac{\tau^{n-1-k}t^{n+k}}{\tau^{2n}-t^{2n}},$$

we obtain (see [2])

$$\begin{aligned}(\widetilde{S}_{n,k}\varphi)(t) &= \frac{1}{\pi i}\int_\Gamma \frac{\tau^{n-1-k}t^k}{\tau^n+t^n}\varphi(\tau)d\tau \\ &= \frac{1}{\pi i}\int_\Gamma \frac{\tau^{2n-1-k}t^k}{\tau^{2n}-t^{2n}}\varphi(\tau)d\tau - \frac{1}{\pi i}\int_\Gamma \frac{\tau^{n-1-k}t^{n+k}}{\tau^{2n}-t^{2n}}\varphi(\tau)d\tau \\ &= (SP_k\varphi)(t)-(SP_{n+k}\varphi)(t). \quad \blacksquare\end{aligned}$$

Lemma 2. [2] *Let $K(\tau,t)$ be a function analytic in D^+ and continuous on $\overline{D^+}$ with respect to each of its variables. Then*

(1) $\int_\Gamma K(\tau,t)\varphi(\tau)d\tau \in X^+$ *for every* $\varphi\in X$.

(2) $\int_\Gamma K(\tau,t)\varphi^+(\tau)d\tau = 0$ *for every* $\varphi^+\in X^+$.

By the same method as in [2, p. 97], using Lemma 2, we can prove the following result.

Lemma 3. *Let $M(\tau,t)$ admit an analytic prolongation in both variables onto D^+ and let $M(\varepsilon_1\tau,t)=M(\tau,\varepsilon_1 t)=M(\tau,t)$, $M(t,t)=0$ for $\tau,t\in\Gamma$. Suppose that the function $(\tau-t)^{-1}[M(\tau,t)-M(t,t)]$ is continuous on $\overline{D^+}$ with respect to each of its variables. Then*

(1) $\phi^+(t)=\int_\Gamma \frac{\tau^{n-1-k}\,t^k}{\tau^n+t^n}M(\tau,t)\varphi(\tau)d\tau\in X^+$ *for every* $\varphi\in X$.

(2) $\phi^+(t)=0$ *for every* $\varphi\in X^+$.

By the same method as in [2, p. 98], we can prove the following result.

Lemma 4. *Suppose that $M(\varepsilon_1\tau,t)=M(\tau,\varepsilon_1 t)=M(\tau,t)$, $M(t,t)=0$ for $\tau,t\in\Gamma$. Then*

$$\mathcal{N}P_j=P_j\mathcal{N}\ \ (j=1,\dots,2n),\ \ \mathcal{M}=\mathcal{N}(P_k-P_{n+k}),$$

where

$$\begin{aligned}N(\tau,t) &= (\tau-t)^{-1}[M(\tau,t)-M(t,t)],\\ (\mathcal{M}\varphi)(t) &= \frac{1}{\pi i}\int_\Gamma \frac{\tau^{n-1-k}\,t^k}{\tau^n+t^n}M(\tau,t)\varphi(\tau)d\tau,\\ (\mathcal{N}\varphi)(t) &= \frac{1}{\pi i}\int_\Gamma N(\tau,t)\varphi(\tau)d\tau.\end{aligned}$$

In the sequel, for every function $a(t) \in X$ we write

$$(K_a\varphi)(t) = a(t)\varphi(t).$$

Lemma 5. [3] *Let $a(t) \in X$ be fixed. Then for every $k, j \in \{1, 2, \dots, 2n\}$ the following identities hold*

$$P_k K_a P_j = K_{a_{kj}} P_j = P_k K_{a_{kj}},$$

where

$$a_{kj}(t) = \frac{1}{2n}\sum_{\nu=1}^{2n} \varepsilon_{\nu+1}^{j-k}\, a(\varepsilon_{\nu+1}t). \tag{4}$$

Now we deal with the equation of the form (1). Rewrite this equation as follows

$$\varphi(t) + \frac{b(t)}{\pi i}\int_\Gamma \frac{\tau^{n-1-k}\, t^k}{\tau^n + t^n}\varphi(\tau)d\tau + \frac{1}{\pi i}\int_\Gamma \frac{\tau^{n-1-k}\, t^k}{\tau^n + t^n}\widetilde{M}(\tau, t)\varphi(\tau)d\tau = f(t), \tag{5}$$

where $\quad b(t) = M(t,t),\ \widetilde{M}(\tau,t) = M(\tau,t) - M(t,t),\ \widetilde{M}(t,t) = 0, \quad t \in \Gamma.$

In the sequel, assume that $\widetilde{M}(\varepsilon_1\tau, t) = \widetilde{M}(\tau, \varepsilon_1 t) = \widetilde{M}(\tau, t)$.

From Lemmas 1 and 4 we can write (5) in the form

$$\varphi(t) + b(t)[S(P_k - P_{n+k})\varphi](t) + [\mathcal{N}(P_k - P_{n+k})\varphi](t) = f(t), \tag{6}$$

where $\quad N(\tau,t) = (\tau - t)^{-1}[\widetilde{M}(\tau,t) - \widetilde{M}(t,t)], (\mathcal{N}\varphi)(t) = \dfrac{1}{\pi i}\displaystyle\int_\Gamma N(\tau,t)\varphi(\tau)d\tau.$

Consider the following system of equations

$$\begin{cases} \varphi_k(t) + \tilde{b}(t)(S\varphi_k)(t) - \tilde{b}_1(t)(S\varphi_{n+k})(t) + (N\varphi_k)(t) = (P_k f)(t), \\ \varphi_{n+k}(t) + \tilde{b}_1(t)(S\varphi_k)(t) - \tilde{b}(t)(S\varphi_{n+k})(t) - (N\varphi_{n+k})(t) = (P_{n+k} f)(t), \end{cases} \tag{7}$$

where $(\varphi_k, \varphi_{n+k}) \in X \times X$ is unknown and $\tilde{b}(t) = b_{kk}(t)$, $\tilde{b}_1(t) = b_{k,n+k}(t)$; $b_{kk}(t)$, $b_{k,n+k}(t)$ are defined by (4).

By the same algebraic method as in [2, p. 103], using Lemma 5 and (3), we can prove the following result:

Lemma 6. *Equation (6) is solvable in X if and only if the system (7) is solvable in $X \times X$. Moreover, every solution of (6) is defined by the formula*

$$\varphi(t) = f(t) - b(t)[S(P_k - P_{n+k})\tilde{\varphi}](t) - [\mathcal{N}(P_k - P_{n+k})\tilde{\varphi}](t),$$

where $\tilde{\varphi}(t) = (P_k\varphi_k)(t) + (P_{n+k}\varphi_{n+k})(t)$ and $(\varphi_k, \varphi_{n+k})$ is a solution of the system (7) in $X \times X$.

In the sequel, assume that $1 - a(t)d(t) \neq 0, t \in \Gamma$, where $a(t) = \tilde{b}(t) + \tilde{b}_1(t)$, $d(t) = \tilde{b}(t) - \tilde{b}_1(t)$.

Denote

$$N_1(\tau, t) = \frac{N(\tau, t)[a(\tau) - d(t)](\tau - t) + d(t)[a(\tau) - a(t)]}{[1 - a(t)d(t)](\tau - t)},$$

$$(\mathcal{N}_1\varphi)(t) = \frac{1}{\pi i}\int_{\Gamma} N_1(\tau, t)\varphi(\tau)d\tau.$$

3. Main Results

Now we can formulate the main results

Theorem 1. *Suppose $N(\tau, t)$ and $N_1(\tau, t)$ are functions which admit an analytic prolongation in D^+ and are continuous on $\overline{D^+}$ with respect to each of their variables. Then equation (6) admits all solutions in a closed form.*

Proof. Due to the results of Lemma 6, it is enough to show that the system (7) admits all solutions in a closed form.

System (7) is equivalent to the following system:

$$\begin{cases} \psi_1(t) + a(t)(S\psi_2)(t) + (\mathcal{N}\psi_2)(t) = g_1(t), \\ \psi_2(t) + d(t)(S\psi_1)(t) + (\mathcal{N}\psi_1)(t) = g_2(t), \end{cases} \tag{8}$$

where

$$\begin{aligned} g_1(t) =& (P_k f)(t) + (P_{n+k} f)(t), \quad g_2(t) = (P_k f)(t) - (P_{n+k} f)(t), \\ \psi_1(t) =& \varphi_k(t) + \varphi_{n+k}(t), \quad \psi_2(t) = \varphi_k(t) - \varphi_{n+k}(t). \end{aligned} \tag{9}$$

Rewrite this system as follows:

$$\begin{cases} \psi_1(t) + (K_a S\psi_2)(t) + (\mathcal{N}\psi_2)(t) = g_1(t), \\ \psi_2(t) - (K_d S K_a S\psi_2)(t) - (K_d S\mathcal{N}\psi_2)(t) - (\mathcal{N}K_a S\psi_2)(t) - (\mathcal{N}^2\psi_2)(t) = g_3(t), \end{cases} \tag{10}$$

where

$$g_3(t) = g_2(t) - d(t)(Sg_1)(t) - (\mathcal{N}g_1)(t).$$

To solve System (10), it is enough to solve the equation

$$\psi_2(t) - (K_d S K_a S\psi_2)(t) - (K_d S\mathcal{N}\psi_2)(t) - (\mathcal{N}K_a S\psi_2)(t) - (\mathcal{N}^2\psi_2)(t) = g_3(t). \tag{11}$$

By our assumption for $N(\tau, t)$, according to Lemma 2, we have

$$\mathcal{N}^2 = 0, \quad S\mathcal{N} = \mathcal{N}, \quad \mathcal{N}S = -\mathcal{N}.$$

Hence, (11) is equivalent to the following equation:

$$\psi_2(t) - (K_d S K_a S \psi_2)(t) + (K_d \mathcal{N} S \psi_2)(t) - (\mathcal{N} K_a S \psi_2)(t) = g_3(t)$$

Rewrite this equation as follows:

$$\psi_2(t) - [1 - d(t)a(t)]^{-1} [(K_d S K_a - K_d K_a S - K_d N + N K_a)(S\psi_2)(t)] = g_4(t)$$

where $g_4(t) = [1 - d(t)a(t)]^{-1} g_3(t)$, i.e.,

$$\psi_2(t) - (\mathcal{N}_1 S \psi_2)(t) = g_4(t). \tag{12}$$

By our assumption for $N_1(\tau, t)$, according to Lemma 2, we have

$$(\mathcal{N}_1 \psi_2^+)(t) = 0, \;\; (\mathcal{N}_1 \psi_2^-)(t) \in X^+,$$

where

$$\psi_2^+(t) = (P\psi_2)(t), \;\; \psi_2^-(t) = -(Q\psi_2)(t).$$

Hence, from (12), we obtain

$$\psi_2^+(t) - (\mathcal{N}_1 \psi_2^-)(t) - (\psi_2^-)(t) = g_4(t). \tag{13}$$

Equation (13) is just a Riemann boundary value problem

$$\phi^+(t) - \phi^-(t) = g_4(t),$$

where

$$\begin{aligned} \phi^+(t) &:= \psi_2^+(t) - (\mathcal{N}_1 \psi_2^-)(t) \in X^+, \\ \phi^-(t) &:= \psi_2^-(t) \in X^-. \end{aligned} \tag{14}$$

This equation has a unique solution

$$\begin{cases} \phi^+(t) = \dfrac{1}{2} g_4(t) + \dfrac{1}{2}(S g_4)(t), \\ \phi^-(t) = -\dfrac{1}{2} g_4(t) + \dfrac{1}{2}(S g_4)(t). \end{cases}$$

From (9), (10), and (14), we obtain

$$\psi_2(t) = \psi_2^+ - \psi_2^- = \phi^+(t) - \phi^-(t) + (\mathcal{N}_1 \phi^-)(t)$$

and

$$\begin{cases} \varphi_k(t) = \dfrac{1}{2}[g_1(t) + \psi_2(t) - a(t)(S\psi_2)(t) - (\mathcal{N}\psi_2)(t)], \\ \varphi_{n+k}(t) = \dfrac{1}{2}[g_1(t) - \psi_2(t) - a(t)(S\psi_2)(t) - (\mathcal{N}\psi_2)(t)]. \end{cases}$$

The theorem is proved. ■

By a similar argument, we prove a dual statement, namely we have

Theorem 2. *Suppose $N(\tau, t)$ and $N_1(\tau, t)$ are functions which admit an analytic prolongation in both variables onto D^- and continuous on $\overline{D^-}$ with respect to each of their variables. Then Eq. (6) admits all solutions in a closed form.*

Theorem 3. *Let $N(\tau, t)$ admit an analytic prolongation in both variables onto D^+ and let $N_1(\tau, t)$ admit an analytic prolongation in τ and meromophic prolongation in t onto D^+. Suppose that $N_1(\tau, t)$ has all poles in variable t at points $z_j (\in D^+)$ of order m_j, respectively ($j = 1, 2, ..., s$). Moreover, $N(\tau, t)$ and $\prod_{j=1}^{s}(t - z)_j^{m_j} N_1(\tau, t)$ are contiuous functions on $\overline{D^+}$ with respect to each of theirs variables. Then the equation (6) admits all solutions in a closed form.*

Proof. By the assumptions, we obtain the representation (see [2])

$$N_1(\tau, t) = [h(t)]^{-1} N_1^+(\tau, t),$$

where $h(t) = \prod_{j=1}^{s}(t - z_j)^{m_j}$, $z_j \in D^+$, $N_1^+(\tau, t)$ admits an analytic prolongation in both variables onto D^+. Hence, (12) has the following form:

$$h(t)\psi_2^+(t) - (\mathcal{N}_1^+\psi_2^-)(t) = h(t)\psi_2^+(t) + h(t)g_4(t),$$

where

$$(\mathcal{N}_1^+\psi)(t) = \frac{1}{\pi i}\int_\Gamma N_1^+(\tau, t)\psi(\tau)d\tau.$$

Since $\phi^+(t) = h(t)\psi_2^+(t) - (\mathcal{N}_1^+\psi_2^-)(t) \in X^+$, $\psi_2^-(t) \in X^-$, the last equation is a Riemann boundary problem, and from this equation, we can find every solution of (9) in a closed form. ■

Acknowledgement. The author is greatly indebted to Professor Ng. V. Mau for valuable advice and various suggestions that led to the improvement of this work.

References

1. F. D. Gakhov, *Boundary Value Problems*, Oxford, 1996; 3rd ed., Nauka, Moscow, 1977 (Russian).
2. Ng. V. Mau, *Generalized Algebraic Elements and Linear Singular Integral Equations with Transformed Arguments*, WPW, Warszawa, 1989.
3. Ng. V. Mau and Ng. M. Tuan, On solutions of integral equations with analytic kernel and rotations, *Annales Polonici Mathematici* **LXIII** (3) (1996) 293-300.
4. D. Prezeworska–Rolewicz, *Equations with Transformed Arguments, An Algebraic Approach*, Elsevier Scientific Publishing, Amsterdam and PWN, Warzawa, 1973.
5. D. Prezeworska–Rolewicz and S. Rolewicz, *Equations in Linear Spaces*, Monograph. Math., Vol. 47, PWN, Warzawa, 1968.