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# On the Convergence of Random Mappings*

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**Abstract.** In this paper, two types of convergence of random mapping are defined and the relation between them is discussed. More results are obtained for the case of random operators and the case of random mappings based on random integrals.

## 1. Introduction

Let $(X, d)$ be a complete separable metric space and $Y$ be a separable Banach space. By definition, a deterministic mapping from $X$ into $Y$ is a rule that assigns to each element $x \in X$ a unique element $\Phi x \in Y$, which is called the image of $\Phi$ under $x$. Due to errors in the measurements and inherent randomness of the environment, the image $\Phi x$ is not known exactly. Therefore, instead of considering $\Phi x$ as an element of $Y$ we have to think of it as a random variable with values in $Y$.

By a random mapping from $X$ into $Y$ we mean a rule $\Phi$ that assigns to each element $x \in X$ a unique $Y$-valued random variable $\Phi x$. Random mappings can be regarded as a "random generalization" of deterministic random mappings and also arise naturally as a generalization of stochastic processes and random fields. This is one of the basic concepts in the theory of Random Dynamical System in an infinite dimensional space (see [1] and references therein). If $X$ is a Banach space, a random mapping is said to be a random operator if the mapping $x \mapsto \Phi x$ is linear and continuous in probability. Some aspects of random operators in Banach space were discussed in [8-10, 12].

The purpose of the paper is to study the convergence of random mappings from $X$ into $Y$. In Sec. 2 we shall define two types of convergence (the convergence in probability and the convergence in law) and discuss the relation between

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them. In Sec. 3 we deal with the convergence of random operators. The main result of this section is Theorem 3.2 which claims that the limit in law of a sequence of random operators is again a random operator. At the end of the paper the convergence of random operators generated by a random integral is investigated.

## 2. Convergence of Random Mappings

First let us give some basic definitions and a few examples of random mappings.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space, $E$ a separable Banach space with the dual space $E^*$ and $\mathcal{B}(E)$ its Borel $\sigma$-algebra. A measurable mapping $u$ from $(\Omega, \mathcal{F})$ into $(E, \mathcal{B}(E))$ is called a $E$-valued random variable. The set of all real-valued random variables (r.v.'s for short) and the set of all $E$-valued random variables are denoted by $L_0(\Omega)$ and $L_0^E(\Omega)$, respectively. We do not distinguish two $E$-valued random variables which are equal almost surely. The set $L_0^E(\Omega)$ equipped with the topology of convergence in probability becomes a $F$-space (complete metric linear space). For $p \geq 1$, $L_p^E(\Omega)$ stands for the Banach space of $E$-valued random variable $u$ with $\|u\| = (E\|u\|^p)^{1/p} < \infty$. For each $u \in L_0^E(\Omega)$ the law of $u$ is denoted by $\mathcal{L}(u)$. The characteristic function of a probability measure on $S$ denoted by $\hat{\mu}$ is a mapping from $E^*$ into $C$ given by

$$\hat{\mu}(y^*) = \int_E \exp\{i\langle x, y*\rangle\} d\mu(x).$$

The characteristic function of an $E$-valued random variable is defined as the characteristic function of its law. If the sequence $(u_n)$ of $L_0^E(\Omega)$ converges in probability to $u$, then we write $P - \lim_n u_n = u$. If $\mathcal{L}(u_n)$ converges weakly to $\mathcal{L}(u)$ we say that $(u_n)$ converges to $u$ in law and write $\mathcal{L} - \lim_n u_n = u$. Finally, as usual $N$ and $N^*$ stand for the set of non-negative integers and the set of positive integers, respectively.

Throughout this paper, $(X, d)$ is a separable metric space and $Y$ is a separable Banach space with the dual space $Y^*$.

**Definition 2.1.** *A family $\Phi = \{\Phi x\}_{x \in X}$ of $Y$-valued r.v.'s indexed by the parameter set $X$ is called a random mapping from $X$ into $Y$ or an $Y$-valued random mapping on $X$. Mathematically, a random mapping $\Phi$ from $X$ into $Y$ is simply a mapping $\Phi : X \to L_0^Y(\Omega)$.*

If $X$ is a time set (i.e., a subset of the real line $R$), then $\Phi$ is called a $Y$-valued stochastic process on $X$. If $X$ is a domain in $R^k$, then $\Phi$ is called a $Y$-valued random field on $X$. Sometimes we refer to $X$ as the parameter set and to $Y$ as the state space.

We define the finite dimensional distribution of a random mapping as follows. Let $x_1, x_2, \ldots, x_k$ be elements of $X$. We define the law of $(\Phi x_1, \ldots, \Phi x_k)$ by

$$P^{\Phi}_{x_1,\ldots,x_k}(A) = P\{\omega : (\Phi x_1, \ldots, \Phi x_k) \in A\}$$

for each $A \in \mathcal{B}(Y^k)$.

The probability measure $P^{\Phi}_{x_1,\ldots,x_k}$ on $Y^k$ is called the finite dimensional distribution of $\Phi$.

For a sequence $\{\Phi_n\}$ of random mappings from $X$ into $Y$, we introduce two types of convergence.

**Definition 2.2.**

(1) *The sequence $\{\Phi_n\}$ is said to converge in probability if, for each $x \in X$, the sequence $\{\Phi_n x\}$ converges in probability. In this case we can define a new random mapping by*

$$\Phi x = P - \lim_{n\to\infty} \Phi_n x.$$

*$\Phi$ is called the limit in probability of $\{\Phi_n\}$ and we write*

$$\Phi = P - \lim_n \Phi_n.$$

(2) *The sequence $\{\Phi_n\}$ is said to converge in law if, for each $k \in N^*$ and for each finite set $(x_1, x_2, \ldots, x_k)$ in $X$, the sequence $\{P^{\Phi_n}_{x_1,\ldots,x_k}\}$ converges weakly as $n \to \infty$.*

**Theorem 2.3.** *If the sequence $\{\Phi_n\}$ converges in law, then there exists a random mapping $\Phi$ such that, for each finite set $(x_1, x_2, \ldots x_k)$ in $X$, the sequence $\{P^{\Phi_n}_{x_1,\ldots,x_k}\}$ converges weakly to $P^{\Phi}_{x_1,\ldots,x_k}$. In this case, $\Phi$ is called the limit in law of the sequence $\{\Phi_n\}$ and we write*

$$\Phi = \mathcal{L} - \lim_n \Phi_n.$$

*Proof.* For each finite set $I = \{(x_1, y_1^*), \ldots, (x_k, y_k^*)\}$ where $x_1, \ldots x_k \in X$, $y_1^*, \ldots y_k^* \in Y^*$, let $P_I^n$ be the law of $\{\langle \Phi x_1, y_1^*\rangle, \ldots, \langle \Phi x_k, y_k^*\rangle\}$. By the assumption, the sequence $\{P_I^n\}$ converges weakly to some probability measure $\mu_I$ on $R^k$. It is easy to check that the family $\{\mu_I\}$ is consistent. Then by the Kolmogorov's celebrated theorem, there exists a random function $B(x, y^*)$ on $X \times Y^*$ whose finite dimensional distributions are the family $\{\mu_I\}$. Fix $x \in X$ and define the mapping $F_x : Y^* \to L_0(\Omega)$ by

$$F_x(y^*) = B(x, y^*).$$

We claim the $F_x$ is linear. Indeed, since $\{\langle \Phi_n x, y_1^* + y_2^*\rangle, \langle \Phi_n x, y_1^*\rangle, \langle \Phi_n x, y_2^*\rangle\}$ converges in law to $\{F_x(y_1^* + y_2^*), F_x(y_1^*), F_x(y_2^*)\}$, the characteristic function $\phi(t)$ of $F_x(y_1^* + y_2^*) - F_x(y_1^*) + F_x(y_2^*)$ is

$$\phi(t) = \lim_n E \exp\{t\langle \Phi_n x, y_1^* + y_2^*\rangle - t\langle \Phi_n x, y_1^*\rangle - t\langle \Phi_n x, y_2^*\rangle\} = 1.$$

This proves that $F_x(y_1^* + y_2^*) - F_x(y_1^*) - F_x(y_2^*) = 0$ a.s. Similarly, $F_x(\lambda y^*) = \lambda F_x(y^*)$ for each $\lambda \in R$. Hence, $F_x$ induces a cylindrical measure $\Gamma$ on $Y$. Actually, $\Gamma$ is a probability measure. Indeed, the characteristic function of $\Gamma$ is

$$\hat{\Gamma}(y^*) = E \exp\{iF_x(y^*)\}. \tag{1}$$

Since $\langle \Phi_n x, y^* \rangle$ converges in law to $F_x(y^*)$, we get

$$\lim_n E \exp\{i\langle \Phi_n x, y^* \rangle\} = E \exp\{iF_x(y^*)\}. \tag{2}$$

Suppose $\mathcal{L}(\Phi_n x)$ converges weakly to the probability measure $\mu$. Then

$$\hat{\mu}(y^*) = \lim_n E \exp\{i\langle \Phi_n x, y^* \rangle\}. \tag{3}$$

From (1) - (3) we obtain $\hat{\Gamma}(y^*) = \hat{\mu}(y^*)$. Hence $\Gamma = \mu$ as claimed. By Theorem 4.6.2 in [5] there exists a unique $Y$-valued random variable denoted by $\Phi x$ such that

$$F_x(y^*) = \langle \Phi x, y^* \rangle \quad \text{for all } y^* \in Y^*.$$

The mapping $x \mapsto \Phi x$ defines a random mapping $\Phi$ from $X$ into $Y$. To complete the proof it remains to prove that the limit of the sequence $\{P^{\Phi_n}_{x_1,\dots,x_k}\}$ is nothing but $P^{\Phi}_{x_1,\dots,x_k}$. Indeed

$$\begin{aligned}
\lim_n \widehat{(P^{\Phi_n}_{x_1,\dots,x_k})}(y_1^*, \dots, y_k^*) &= \lim_n E \exp\Big\{i\sum_{j=1}^{k} \langle \Phi_n x_j, y_j^* \rangle\Big\} \\
&= E \exp\Big\{i\sum_{j=1}^{k} B(x_j, y_j^*)\Big\} \\
&= E \exp\Big\{i\sum_{j=1}^{k} \langle \Phi x_j, y_j^* \rangle\Big\} \\
&= \widehat{(P^{\Phi}_{x_1,\dots,x_k})}(y_1^*, \dots, y_k^*).
\end{aligned}$$

Hence, the sequence $\{P^{\Phi_n}_{x_1,\dots,x_k}\}$ converges weakly to $P^{\Phi}_{x_1,\dots,x_k}$. ∎

**Definition 2.4.** *Two random mapping $\Phi$ and $\Psi$ is called equivalent if they have the same finite dimensional distributions, i.e.,*

$$P^{\Phi}_{x_1,\dots,x_k} = P^{\Psi}_{x_1,\dots,x_k}$$

*for any $x_1, \dots, x_k$ of $X, k \in N^*$.*

In this case, we write $\Phi \overset{\mathcal{L}}{=} \Psi$.

Now we study the relation between two types of convergence. Clearly, if the sequence $\{\Phi_n\}$ converges in probability, then it also converges in law. Under some assumptions, the converse is true if the sequence $\{\Phi_n\}$ is replaced by an equivalent one.

**Definition 2.5.** *A random mapping $\Phi$ is said to be stochastically continuous if the mapping $\Phi : X \to L_0^Y(\Omega)$ is continuous.*

**Theorem 2.6.** *Let $\{\Phi_n\}_{n=1}^{\infty}$ be a sequence of stochastically continuous random mappings converging in law to the stochastically continuous random mapping $\Phi_0$. Then there exist random mappings $\Psi_n$ $(n \in N)$ such that, for each $n \in N$, $\Phi_n \overset{\mathcal{L}}{=} \Psi_n$ and the sequence $\{\Phi_n\}_{n=1}^{\infty}$ converges in probability to $\Psi_0$.*

*Proof.* Let $D = (z_k)_{k=1}^{\infty}$ be the countable set dense in $X$. For each $n \in N$, define $Y^N$-valued random variable $\xi_n$ by

$$\xi_n = (\Phi_n z_i)_{i=1}^{\infty}.$$

Since $\Phi_n$ converges to $\Phi_0$ in law, $\xi_n$ also converges to $\xi_0$ in law. By the Skorokhod theorem [5], there exist $Y^N$-valued random variables $\eta_n = (\eta_{ni})_{i=1}^{\infty}$ such that

$$\mathcal{L}(\eta_n) = \mathcal{L}(\xi_n), \quad \text{for each } n \in N, \tag{4}$$

and

$$P - \lim_n \eta_n = \eta_0. \tag{5}$$

Now fix $n \in N$. Define a random mapping $\Psi_n$ from $X$ into $Y$ as follows. At first, $\Psi_n$ is defined on $D$ by

$$\Psi_n z_i = \eta_{n_i}, \quad (i \in N^*).$$

We claim that $\Psi_n$ can be extended over the entire space $X$. Indeed let $x \in X$ and $(z_{k_i})$ be a subsequence of $D$ converging to $x$. By (4) we have

$$P\left\{\|\Psi_n z_{k_i} - \Psi_n z_{k_j}\| > t\right\} = P\left\{\|\Phi_n z_{k_i} - \Phi_n z_{k_j}\| > t\right\}.$$

Since $\Phi$ is stochastically continuous, $(\Phi_n z_{k_i})$ converges in probability to $\Phi_n x$ as $i \to \infty$. Hence $P - \lim_i \Psi_n z_{k_i}$ exists and the limit denoted by $\Psi_n x$ does not depend on the choice of the approximating sequence $(z_{k_i})$. Now we shall show that

$$\Phi_n \overset{\mathcal{L}}{=} \Psi_n.$$

Indeed let $x_1, ..., x_k$ be elements in $X$. For each $i = 1, \ldots, k$ choose a subsequence $(z_{i_m})$ such that $z_{i_m}$, converges to $x_i$ as $m \to \infty$. Since $\Phi_n z_{i_m} \to \Phi_n x_i$ and $\Psi_n z_{i_m} \to \Psi_n x_i$ in probability as $m \to \infty$, $(\Phi_n z_{1_m}, \ldots, \Phi_n z_{k_m})$ converges in law to $(\Phi_n x_1, \ldots, \Phi_n x_k)$ and $(\Psi_n z_{1_m}, \ldots, \Psi_n z_{k_m})$ converges in law to $(\Psi_n x_1, \ldots, \Psi_n x_k)$. From (4) we obtain

$$\mathcal{L}(\Phi_n x_1, \ldots, \Phi_n x_k) = \mathcal{L}(\Psi_n x_1, \ldots, \Psi_n x_k).$$

In particular, $\{\Psi_n\}$ is also stochastically continuous for each $n \in N$. Now it remains to prove that the sequence $\{\Psi_n\}$ converges in probability to $\Psi_0$. Fix $x \in X$ and let $t > 0$ and $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$P\left\{\|\Psi_0 x - \Psi_0 x'\| > \frac{t}{3}\right\} < \frac{\epsilon}{2}$$

whenever $d(x, x') < \delta$.

Choose $z \in D$ such that $d(z, x) < \delta$. Then

$$P\{\|\Psi_n x - \Psi_0 x\| \geq t\} \leq P\left\{\|\Psi_n x - \Psi_n z\| \geq \frac{t}{3}\right\} + P\left\{\|\Psi_n z - \Psi_0 z\| \geq \frac{t}{3}\right\} + P\left\{\|\Psi_0 z - \Psi_0 x\| \geq \frac{t}{3}\right\}. \tag{6}$$

Since $(\Psi_n x, \Psi_n z)$ converse in law to $(\Psi_0 x, \Psi_0 z)$ and $P - \lim_n \Psi_n z = \Psi_0 z$, we have

$$\overline{\lim_n} P\left\{\|\Psi_n x - \Psi_n z\| \geq \frac{t}{3}\right\} \leq P\left\{\|\Psi_0 x - \Psi_0 z\| \geq \frac{t}{3}\right\},$$

and

$$\lim_n P\left\{\|\Psi_n z - \Psi_0 z\| \geq \frac{t}{3}\right\} = 0.$$

Consequently, from (6) we get

$$\overline{\lim_n} P\{\|\Psi_n x - \Psi_0 x\| \geq t\} \leq 2P\left\{\|\Psi_0 x - \Psi_0 z\| \geq \frac{t}{3}\right\} < \epsilon.$$

Leting $\epsilon \to 0$, we get

$$\lim_n P\{\|\Psi_n x - \Psi_0 x\| \geq t\} = 0.$$

The theorem is fully proved. ■

**Definition 2.7.** *Let $\mathcal{H}$ be a family of random mappings from $X$ into $Y$. We say that $\mathcal{H}$ is stochastically equicontinuous at $x_0$ if $\forall t > 0$, $\forall \epsilon > 0$, $\exists \delta > 0$ such that for each $\Phi \in \mathcal{H}$ we have*

$$P\{\|\Phi x - \Phi x_0\| > t\} < \epsilon$$

*whenever $d(x, x_0) < \delta$.*

*The family $\mathcal{H}$ is said to be stochastically equicontinuous on $X$ if it is stochastically equicontinuous at each point $x \in X$.*

**Theorem 2.8.** *Let $\{\Phi_n\}$ be a sequence of random mappings converges in law to a random mapping $\Phi$. If the sequence $\{\Phi_n\}$ is stochastically equicontinuous at $x_0$, then $\Phi$ is stochastically continuous at $x_0$.*

*In particular if the sequence $\{\Phi_n\}$ is stochastically equicontinuous on $X$, then the random mapping $\Phi$ is stochastically continuous.*

*Proof.* Given $\epsilon > 0$ and $t > 0$. By the stochastic equicontinuity of the sequence $\{\Phi_n\}$ at $x_0$ there exists $\delta > 0$ such that if $d(x, x_0) < \delta$. Then

$$P\{\|\Phi_n x - \Phi_n x_0\| > t\} < \epsilon \quad \text{for all } n \in N^*.$$

Since $(\Phi_n x, \Phi_n x_0)$ converges in law to $(\Phi x, \Phi x_0)$, we have

$$P\{\|\Phi x - \Phi x_0\| > t\} \leq \underline{\lim}_n P\{\|\Phi_n x - \Phi_n x_0\| > t\} \leq \epsilon$$

whenever $d(x, x_0) < \delta$. This proves the theorem. ■

## 3. Convergence of Random Operators

In this section, we restrict ourselves to an important class of random mappings called the class of random operators, whose prototypes are stochastic integrals.

**Definition 3.1.** *Let $X$ be a separable Banach space. A random mapping $\Phi$ from $X$ into $Y$ is called a random operator if $\Phi(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1\Phi(x_1) + \lambda_2\Phi(x_2)$ a.s. $\forall\lambda_1, \lambda_2 \in R$, $\forall x_1, x_2 \in X$, $\lim_{x\to 0} P\{\|x\| > \epsilon\} = 0$ $\forall\epsilon > 0$.*

Note that the exceptional set can depend on $\lambda_1$, $\lambda_2$, $x_1$, $x_2$.

In the case of random operators it is interesting that the assertion of Theorem 2.8 holds without the assumption about the stochastic equicontinuity of the sequence $\{\Phi_n\}$, namely we have the following:

**Theorem 3.2.** *Let $\{\Phi_n\}$ be a sequence of random operators converges in law to a random mapping $\Phi$. Then $\Phi$ is also a random operator.*

*Proof.* At first we shall show that $\Phi : X \to L_0^Y(\Omega)$ is linear. Let $x_1, x_2 \in X$. Because $\{\Phi_n(x_1+x_2), \Phi_n x_1, \Phi_n x_2\}$ converges in law to $\{\Phi(x_1+\Phi x_2), \Phi x_1, \Phi x_2\}$, it follows that, for all $\epsilon > 0$,

$$P\{\|\Phi(x_1 + x_2) - \Phi x_1 - \Phi x_2\| > \epsilon\} \leq \\ \underline{\lim}_n P\{\|\Phi_n(x_1 + x_2) - \Phi_n x_1 - \Phi_n x_2\| > \epsilon\} = 0,$$

i.e., $\Phi(x_1 + x_2) = \Phi x_1 + \Phi x_2$ a.s. Similarly $\Phi(\lambda x) = \lambda\Phi x$ a.s. for all $\lambda \in R$. It remains to prove that $\Phi : X \to L_0^Y(\Omega)$ is continuous. At first we claim that for each $x \in X$, the set $(\Phi_n x)_{n=1}^{\infty}$ is bounded in $L_0^Y(\Omega)$. Indeed let

$$V(t,\epsilon) = \{g \in L_0^Y(\Omega) : P\{\|g\| > t\} < \epsilon\}$$

be a neighborhood of zero. Since $\mathcal{L}(\Phi_n x)$ converges weakly to $\mathcal{L}(\Phi x)$ by Prokhorov's theorem, there exists $T > 0$ such that $P\{\|\Phi_n x\| > T\} < \epsilon$ for all $n$. Let $|r| < \frac{t}{T}$. Then, for each $n \in N^*$,

$$P\{\|r\Phi_n x\| > t\} = P\left\{\|\Phi_n x\| > \frac{t}{|r|}\right\} \leq P\{\|\Phi_n x\| > T\} < \epsilon,$$

that means $r\Phi_n x \in V(t,\epsilon)$ as claimed.

By the principle of uniformly boundedness of linear continuous mappings between two F-spaces [7, Theorem 2.2.1], there exists $\delta > 0$ such that if $\|x\| < \delta$, then $\Phi_n x \in V(t,\epsilon)$, i.e., $P\{\|\Phi_n x\| > t\} < \epsilon$ for all $n$. Consequently

$$P\{\|\Phi x\| > t\} \leq \underline{\lim}_n P\{\|\Phi_n x\| > t\} < \epsilon$$

whenever $\|x\| < \delta$. This proves that $\Phi$ is stochastically continuous. ∎

**Corollary 3.3.** *Let $\{\Phi_n\}_{n=1}^{\infty}$ be a sequence of random operators converging in law. Then there exists a sequence $\{\Psi_n\}_{n=1}^{\infty}$ of random operators such that, for each $n$, $\Phi_n \stackrel{\mathcal{L}}{=} \Psi_n$ and $\{\Psi_n\}_{n=1}^{\infty}$ converges in probability.*

Finally we study the convergence of random operators generated by a random integral. Let $(S, \mathcal{S}, \mu)$ be a finite measurable space. A mapping $M$ :

$\mathcal{S} \to L_0(\Omega)$ is said to be a symmetric Gaussian random measure with the control measure $\mu$ if

(1) For each $A \in \mathcal{S}$, $M(A)$ is a real-valued Gaussian random variable with mean zero and the variance $\mu(A)$.

(2) For every sequence $(A_i)$ of disjoint sets in $\mathcal{S}$ the random variables $M(A_1)$, $M(A_2)$,... are independent and

$$M\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} M(A_n) \quad \text{a.s.}$$

The random integral of Banach space valued functions with respect to random measure $M$ is constructed as follow (see [4, 6]). Let $E$ be a separable Banach space. For a simple function $f: S \to E$, $f = \sum_{i=1}^{n} x_i \lambda_{A_i}$, where $(A_i)$ are disjoint sets, we define

$$\int_S f dM = \sum_{i=1}^{n} x_i M(A_i).$$

A measurable deterministic function $f : S \to E$ is said to be $M$-integrable if there exists simple functions $(f_n)$ such that $f_n$ converges to $f$ in $\mu$-measure and the sequence $\{\int_S f_n\, dM\}$ converges in probability. If $f$ is $M$-integrable, then we put

$$\int_S f\, dM = P-\lim \int_S f_n\, dM.$$

It was shown that this value does not depend on the choice of the approximating sequence $(f_n)$. Among crucial properties of this stochastic integral is the following theorem, which will be used later.

**Theorem 3.4** [4].

(1) *If $f$ is $M$-integrable then $\int_S f\, dM$ is an $E$-valued centered Gaussian random variable with the characteristic function*

$$F(y^*) = \exp\Big\{ - \int_S |\langle f(t), y^* \rangle|^2 d\mu(t) \Big\}.$$

(2) *If $E$ is a Banach space of type 2, then each function $f \in L_2^E(S)$ is $M$-integrable. In this case there exists a constant $C$ depending only on $E$ such that*

$$E\Big\| \int_S f\, dM \Big\|^2 \le C \int_S \|f\|^2 d\mu \quad \textit{for each } f \in L_2^E(S).$$

Recall that a Banach space $E$ is said to be of type 2 if for each sequence $(y_n) \in E$ such that $\sum_{n=1}^{\infty} \|y_n\|^2 < \infty$ the series $\sum_{n=1}^{\infty} \alpha_n y_n$ converges a.s., where $(\alpha_n)$ is the sequence of independent identical distributive $N(0,1)$ random variables.

**Theorem 3.5.** *Let $(M_n)_{n=0}^{\infty}$ be symmetric Gaussian random measures with the control measures $(\mu_n)_{n=0}^{\infty}$, respectively on the measurable space $(S, \mathcal{S})$ and let $Y$*

*be a Banach space of type 2. Denoted by $C_b[S,Y]$ the Banach space of bounded continuous functions on $S$ taking values in $Y$. Define*

$$\Phi_n f = \int_S f\, dM_n.$$

*Then*
(1) *$\Phi_n$ is a random mapping from $C_b[S,Y]$ into $Y$ for each $n \in N$.*
(2) *If $\mu_n$ converges weakly to $\mu_0$ then $\Phi_n$ converges in law to $\Phi_0$.*

*Proof.* (1) It follows easily from Theorem 3.4 and Chebyshev's inequality.
(2) For the proof of this claim we need the following lemma

**Lemma 3.6.** ([3, Theorem 4.2 Chap. 1]) *Let $E$ be a separable Banach space and $(\xi_n)_{n=0}^{\infty}$ be a sequence of $E$-valued random variables such that, for each $m \in N^*$, there exists a sequence $(\xi_{n,m})$ of $E$-valued random variables satisfying*

$$\mathcal{L} - \lim_n \xi_{n,m} = \xi_m,$$
$$\mathcal{L} - \lim_m \xi_m = \xi_0,$$
$$\lim_m \overline{\lim_n} P\{\|\xi_{n,m} - \xi_n\| > \epsilon\} = 0 \quad \forall \epsilon > 0.$$

*Then $\mathcal{L} - \lim_n \xi_n = \xi_0$.*

Let $(f_i)_{i=1}^k$ be elements of $C_b[S,Y]$. Put $h = (f_1, \dots, f_k) : S \to Y^k$. We have to show that $\int_S h\, dM_n = \left(\int_S f_i\, dM_n\right)_{i=1}^k$ converges in law to $\int_S h\, dM_0 = \left(\int_S f_i\, dM_0\right)_{i=1}^k$. Since $\mu_n$ converges weakly to $\mu_0$ by Prokhorov's theorem for each $m \in N^*$, we can find a compact set $K \subset S$ such that $\sup_{n \in N} \mu_n(S) = L < \infty$ and $\sup_{n \in N} \mu_n(K^c) < 1/m$. Further we can choose a simple function $h_m = \sum_{i=1}^r y_i \lambda_{A_i}$ such that $\sup_{s \in K} \|h_m(s) - h(s)\| < 1/m$, $\|h_m\| = \sup_{s \in S} \|h_m(s) \leq \sup_{s \in S} \|h(s)\| = \|h\|$ and $(A_i)$ are $\mu_0$-continuous sets. Put

$$\xi_{n,m} = \int_S h_m\, dM_n = \sum_{i=1}^r y_i\, M_n(A_i),$$
$$\xi_m = \int_S h_m\, dM_0 = \sum_{i=1}^r y_i\, M_0(A_i),$$
$$\xi_n = \int_S h\, dM_n,$$
$$\xi_0 = \int_S h\, dM_0.$$

Because $\mu_n(A_i) \to \mu(A_i)$ as $n \to \infty$, we have $\mathcal{L} - \lim_n M_n(A_i) = M_0(A_i)$ $(i = 1, \dots, k)$. By the independent of the sequence $\{M_n(A_i)\}_{i=1}^r$, it follows that $\mathcal{L} - \lim_n \xi_{n,m} = \xi_m$. Next, note that $Y^k$ is also a Banach space of type 2, so by Lemma 3.6, we have

$$E\|\xi_m - \xi_0\|^2 \leq C_1 \int_S \|h_m - h\|^2 d\mu_0 \leq C_1 m^{-1}(L + 4\|h\|^2), \tag{7}$$

$$E\|\xi_{n,m} - \xi_n\|^2 \leq C_1 \int_S \|h_m - h\|^2 d\mu_n \leq C_1 m^{-1}(L + 4\|h\|^2), \tag{8}$$

where $C_1$ is some constant.

Using Chebyshev's inequality from (7) and (8), we get

$$P - \lim_m \xi_m = \xi_0 ,$$

$$\lim_m \overline{\lim}_n P\{\|\xi_{n,m} - \xi_n\| > \epsilon\} = 0 \ \forall \epsilon > 0.$$

Now the assertion of the theorem follows from Lemma 3.6.

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