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Short Communication

Convergence Rate of Post-Widder Approximate Inversion of the Laplace Transform

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1. Introduction

Let $f \in L_2(\mathbb{R}_+)$. Then the integral

$$F(p) = \int_0^\infty e^{-pt} f(t)dt \tag{1}$$

exists and is called the Laplace transform of f. The Laplace transform occurs frequently in the applications of mathematics, especially in those branches involving solutions of differential equations and convolution integral equations. If the image F is known in the complex plane, the original f can be computed by the Bromwich contour integral [5]

$$f(t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} F(p)e^{pt}dp, \quad d > 0. \tag{2}$$

However, if the image F is known only on the positive axis $\mathbb{R}_+$, one should use the Post-Widder formula instead

$$f(t) = \lim_{n\to\infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} F^{(n)}\left(\frac{n}{t}\right). \tag{3}$$

Formula (3) itself is an approximation scheme for inverting the Laplace transform when the limit is dropped [1]. Jagerman [3] applied this approximation scheme to invert the Laplace transform. However, so far the convergence rate of this approximation scheme has not been studied yet. In this paper we obtain the convergence rate of the Post-Widder approximate inversion of the Laplace transform in some function spaces.

2. Convergence Rate

Let
$$f^*(s) = \int_0^\infty t^{s-1} f(t)dt \tag{4}$$
be the Mellin transform of function f [4]. Applying the Parseval formula for the Mellin convolution [4]
$$\int_0^\infty k(zy)f(y)dy = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} k^*(s)f^*(1-s)z^{-s}ds, \tag{5}$$
valid if $k, f \in L_2(\mathbb{R}_+)$, with $k(y) = \exp(-y), k^*(s) = \Gamma(s)$, we have
$$F(p) = \int_0^\infty \exp(-pt)f(t)dt = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \Gamma(s)f^*(1-s)p^{-s}ds. \tag{6}$$
From (6), one can prove that
$$\frac{(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1} F^{(n)}\left(\frac{n}{t}\right) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{\Gamma(1-s+n)}{\Gamma(1+n)} n^s f^*(s)t^{-s}ds. \tag{7}$$
Applying the Plancherel theorem for the Mellin transform [4] we obtain
$$\left\|\frac{(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1} F^{(n)}\left(\frac{n}{t}\right) - f(t)\right\|_{L_2(\mathbb{R}_+)}$$
$$= (2\pi)^{-1}\left\|\left[\frac{\Gamma(1-s+n)}{\Gamma(1+n)}n^s - 1\right]f^*(s)\right\|_{L_2(1/2-i\infty,1/2+i\infty)} \tag{8}$$
We have
$$\lim_{n\to\infty} \frac{\Gamma(1-s+n)}{\Gamma(1+n)}n^s = 1,$$
and
$$\left|\frac{\Gamma(1-s+n)}{\Gamma(1+s)}n^s\right| \le \left|\frac{\Gamma(1-\Re e s+n)}{\Gamma(1+n)}n^{\Re e s}\right| \le C,$$
where constant C does not depend on n and $\mathcal{I}m\, s$ (see [5]). Hence, the right-hand side of (8) tends to 0 as $n \to \infty$, and so does the left-hand side of (8). Therefore, formula (3) is valid, if the limit is understood in $L_2(\mathbb{R}_+)$ norm. Consequently, function $f(t)$ when $t \in (0, T)$ can be recovered from function $F(p)$ on any half-line (p, ∞). In particular,
$$\|f(t)\|_{L_2(0,T)} = \lim_{n\to\infty}\left\|\frac{p^n}{n!}F^{(n)}(p)\right\|_{L_2(n/T,\infty)}, \tag{9}$$
for all positive $T, 0 < T \le \infty$.

Lemma. *Let x and y be real numbers. Then*
$$\frac{\Gamma(x+iy)}{\Gamma(x)}x^{-iy} = 1 + O\left(\frac{y^2+1}{x}\right), \tag{10}$$

as x tends to ∞.

Proof. We have [2]

$$|\Gamma(x+iy)| \leq \Gamma(x).$$

Therefore, the inequality (10) should be proved only for y being small in absolute value in comparison with $\sqrt{x}$, say, $|(1+y^2)/x| < 1/2$. Using the Stirling's asymptotic formula for the gamma function [2]

$$\Gamma(z) = \sqrt{2\pi} e^{-z} z^{z-1/2}\left(1 + O\left(\frac{1}{z}\right)\right),$$

valid for large $|z|$ in the domain $|\arg z| < \pi$, we obtain

$$\frac{\Gamma(x+iy)}{\Gamma(x)} x^{-iy} = e^{-iy}\left(1 + \frac{iy}{x}\right)^{x+iy-1/2}\left(1 + O\left(\frac{1}{x}\right)\right). \tag{11}$$

We have

$$\begin{aligned}
&e^{-iy}\left(1+\frac{iy}{x}\right)^{x+iy-1/2}\left(1+O\left(1/x\right)\right)\\
&= \exp(-iy + (x+iy-1/2)\ln(1+iy/x))\left(1+O\left(\frac{1}{x}\right)\right)\\
&= \exp(-iy + (x+iy-1/2)(iy/x + O(y^2/x^2))\left(1+O\left(\frac{1}{x}\right)\right)\\
&= \exp(O(y^2/x))\left(1+O\left(\frac{1}{x}\right)\right) = \left(1+O\left(\frac{y^2}{x}\right)\right)\left(1+O\left(\frac{1}{x}\right)\right)\\
&= 1 + O\left(\frac{1+y^2}{x}\right).
\end{aligned} \tag{12}$$

The lemma is thus proved. ∎

Now, let the real part of s be $1/2$. Remembering that [2]

$$\frac{\Gamma(a+x)}{\Gamma(b+x)} x^{b-a} = 1 + O\left(\frac{1}{x}\right),$$

$$\frac{\Gamma(a+n-s)}{\Gamma(a+n)} n^s = \frac{\Gamma(a+n-s)}{\Gamma(a+n-1/2)} n^{s-1/2} \frac{\Gamma(a+n-1/2)}{\Gamma(a+n)} n^{1/2},$$

we get

$$\begin{aligned}
\frac{\Gamma(a+n-s)}{\Gamma(a+n)} n^s &= \left(1 + O\left(\frac{1+(\mathcal{I}m s)^2}{a+n-1/2}\right)\right)\left(1+O\left(\frac{1}{n}\right)\right)\\
&= 1 + O\left(\frac{s^2}{n}\right).
\end{aligned} \tag{13}$$

Using relation (13) and equation (8) we have

$$\begin{aligned}
&\left\|\frac{(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1} F^{(n)}\left(\frac{n}{t}\right) - f(t)\right\|_{L_2(\mathbb{R}_+)}\\
&\leq \frac{C}{n}\|s^2 f^*(s)\|_{L_2(1/2-i\infty,1/2+i\infty)}.
\end{aligned} \tag{14}$$

If f is twice differentiable, and moreover, $t^2(d^2f(t)/dt^2) \in L_2(\mathbb{R}_+)$, then

$$s^2 f^*(s) \in L_2(1/2 - i\infty, 1/2 + i\infty),$$

and the norm in the right hand-side of inequality (14) is finite. Since $s^2/n = O(s/\sqrt{n})$ when $|s^2/n| < 1$, the relation

$$\frac{\Gamma(a+n-s)}{\Gamma(a+n)} n^s = 1 + O\left(\frac{s}{\sqrt{n}}\right) \tag{15}$$

is also valid. Hence,

$$\left\| \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} F^{(n)}\left(\frac{n}{t}\right) - f(t) \right\|_{L_2(\mathbb{R}_+)} \leq \frac{C}{\sqrt{n}} \|s\, f^*(s)\|_{L_2(1/2-i\infty, 1/2+i\infty)}. \tag{16}$$

Therefore, if we weaken the smooth condition on f (one time less differentiable), then the convergence rate will be of order $O(1/\sqrt{n})$ instead of $O(1/n)$. Thus, we obtain

Theorem.
(a) *Let $f \in L_2(\mathbb{R}_+)$ be differentiable and $t(df(t)/dt) \in L_2(\mathbb{R}_+)$. Then $[(-1)^n/n!]$ $(n/t)^{n+1}F^{(n)}(n/t)$ converges to $f(t)$ in $L_2(\mathbb{R}_+)$ norm with the rate $n^{-1/2}$.*
(b) *Let $f \in L_2(\mathbb{R}_+)$ be twice differentiable and $t^2(d^2f(t)/dt^2) \in L_2(\mathbb{R}_+)$. Then $[(-1)^n/n!](n/t)^{n+1}F^{(n)}(n/t)$ converges to $f(t)$ in $L_2(\mathbb{R}_+)$ norm with the rate n^{-1}.*

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