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Eventual Stability in Terms of Two Measures of Nonlinear Differential Systems*

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Abstract. In this paper, we investigate eventual stability in terms of two measures of nonlinear differential systems. We obtain some sufficient conditions by using Lyapunov direct method and some comparison theorems. Some examples are also worked out.

1. Introduction

We have studied stability of the solutions of nonlinear differential systems, and learned that Lyapunov stability, especially uniform stability and uniform asymptotic stability, plays an important role in a physical system. However, we sometimes only need to study the ultimate state of the solution. This kind of stability is called *eventual stability* which we shall define in the next section.

There are several different concepts of stability studied in the literature, such as eventual stability, partial stability, conditional stability, etc. To unify these varieties of stability notions and to offer a general basis for investigation, it is convenient to introduce stability in terms of two mearsures.

In this paper we will investigate eventually uniform stability and eventually uniform asymptotical stability in terms of two measures. We will obtain some sufficient conditions based on Lyapunov function. Furthermore, by using differential inequalities, we will establish some comparison theorems. Our results improve some of the earlier findings and may be suitable for many applications.

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2. Preliminaries

Consider the differential system

$$x' = f(t,x), \qquad x(t_0) = x_0, \tag{2.1}$$

where $f \in C[R_+ \times R^n, R^n]$.

Let us begin by defining the following classes of functions for future use.
$K = \{a \in C[R_+, R_+] : a(u) \text{ is strictly increasing and } a(0) = 0\}$,
$\Gamma = \left\{mh \in C[R_+ \times R^n, R_+] : \inf_{(t,x)\in R_+\times R^n} h(t,x) = 0\right\}$.

Definition 2.1. *Let h_0, $h \in \Gamma$, then we say that h_0 is uniformly finer than h if there exist a $\delta > 0$ and a function $\varphi \in K$ such that*

$$h_0(t,x) < \delta \ \textit{ implies } \ h(t,x) < \varphi(h_0(t,x)).$$

Let h_0, $h \in \Gamma$. Now we can define the eventually uniform stability and eventually uniform asymptotical stability. Let $x(t) = x(t, t_0, x_0)$ be a solution of (2.1).

Definition 2.2. *The system (2.1) is said to be (h_0, h)-eventually uniformly stable if, for every $\epsilon > 0$, there exist two positive numbers $\delta = \delta(\ \epsilon) > 0, \tau = \tau(\epsilon) > 0$, such that*

$$h(t, x(t)) < \epsilon,\ t \geq t_0 \geq \tau,\ \textit{ provided that } \ h_0(t_0, x_0) < \delta.$$

Definition 2.3. *The system (2.1) is said to be (h_0, h)-eventually quasi-uniformly asymptotically stable if, for every $\epsilon > 0$, there exist positive numbers δ_0, τ_0 and $T = T(\epsilon)$, such that*

$$h(t, x(t)) < \epsilon,\ t \geq t_0 + T,\ t_0 \geq \tau_0,\ \textit{ provided that } \ h_0(t_0, x_0) < \delta_0.$$

Definition 2.4. *The system (2.1) is said to be (h_0, h)-eventually uniformly asymptotically stable if Definitions 2.2 and 2.3 hold together.*

We need the following known result for our discussion.

Lemma 2.1. [1] *Let $g \in C[R_+ \times R, R]$ and $r(t) = r(t, t_0, u_0)$ be the maximal solution of*

$$u' = g(t,u) \qquad u(t_0) = u_0, \tag{2.2}$$

existing on J. Suppose $m \in C[R_+, R_+]$, $D^+m(t) \leq g(t, m(t))$, $t \in J$, where D^+ is Dini derivative. Then

$$m(t_0) \leq u(t_0) \ \textit{ implies } \ m(t) \leq r(t),\ t \in J.$$

Similarly, the set $u = 0$ with respect to the system (2.2) can be defined as eventually uniformly stable and eventually uniformly asymptotically stable.

3. Main Results

In this section we shall state and prove our results. We define for any h_0, $h \in \Gamma$, $S(h,\rho) = \{(t,x) : h(t,x) < \rho\}$, $S^c(h,r) = \{(t,x) : h(t,x) \geq r\}$.

Theorem 3.1. *Assume that*

(i) *h_0 is uniformly finer than h;*

(ii) *there exists a function $V \in C[S(h,\rho), R_+]$, $V(t,x)$ is locally lipschitzian in x, and*

$$b(h(t,x)) \leq V(t,x) \leq a(h_0(t,x)) \text{ for } (t,x) \in S(h,\rho)\bigcap S^c(h_0,r) \text{ and } t \geq \theta(r),$$

where a, $b \in K$ and $\theta(r)$ is continuous and monotonic decreasing in r;

(iii) *$D^+V(t,x) \leq 0$ for $(t,x) \in S(h,\rho)\bigcap S^c(h_0,r)$ and $t \geq \theta(r)$.*

Then the system (2.1) *is (h_0,h)-eventually uniformly stable.*

Proof. From (i), there exist a $\delta^* > 0$ and a function $\varphi \in K$, such that

$$h(t,x) \leq \varphi(h_0(t,x)), \text{ provided that } h_0(t,x) < \delta^*.$$

For every $\epsilon \in (0,\rho)$, choose $\delta = \delta(\epsilon)$ such that $\delta \in (0,\delta^*]$ and $a(\delta) < b(\epsilon), \varphi(\delta) < \epsilon$. Observe that if $h_0(t_0,x_0) < \delta$, then $h(t_0,x_0) \leq \varphi(h_0(t_0,x_0)) < \varphi(\delta) < \epsilon$. Let $\tau = \tau(\epsilon) = \theta(\delta)$, and we can prove that

$$h_0(t_0,x_0) < \delta \text{ implies } h(t,x(t)) < \epsilon, \ t \geq t_0 \geq \tau.$$

If this is not true, then there exist some solution $x(t) = x(t,t_0,x_0)$ of (2.1) with $h_0(t_0,x_0) < \delta$ and t_1, t_2, such that $t_2 > t_1 > t_0 \geq \tau$,

$$h_0(t_1,x(t_1)) = \delta, \ \ h(t_2,x(t_2)) = \epsilon, \ \text{ and } \ (t,x) \in S(h,\epsilon)\bigcap S^c(h_0,\delta), \ t \in [t_1,t_2].$$

From (ii) and (iii), we have

$$\begin{aligned} b(\epsilon) = b(h(t_2,x(t_2))) &\leq V(t_2,x(t_2)) \leq V(t_1,x(t_1)) \\ &\leq a(h_0(t_1,x(t_1))) = a(\delta) < b(\epsilon). \end{aligned}$$

This absurdity shows that the system (2.1) is (h_0,h)-eventually uniformly stable, which we are led to prove. ∎

Theorem 3.2. *Assume* (i) *and* (ii) *of Theorem 3.1 hold and*

(iii) $D^+V(t,x) \le -c(h_0(t,x))$ *for* $(t,x) \in S(h,\rho) \bigcap S^c(h_0,r)$ *and* $t \ge \theta(r)$, *where* $c \in K$;

(iv) $h_0(t,x)$ *is locally lipschitzian in* x *and there exists* $\gamma > 0$, *such that* $D_-h_0(t,x) < 0$ *provided that* $h_0(t,x) = \gamma$.

Then the system (2.1) *is* (h_0,h)*-eventually uniformly asymptotically stable.*

Proof. By Theorem 3.1, the system is (h_0,h)-eventually uniformly stable. Therefore, there exist a $\delta_0 \in (0,\gamma)$ and $\tau_0 > 0$, such that if $h_0(t_0,x_0) < \delta_0$, then $h(t,x(t)) < \rho$, $t \ge t_0 \ge \tau_0$. Let $\epsilon \in (0,\rho)$, designate $\delta = \delta(\epsilon)$, $\tau = \tau(\epsilon) = \theta(\delta)$, and

$$h_0(t_0,x_0) < \delta \text{ implies } h(t,x(t)) < \epsilon,\ t \ge t_0 \ge \tau.$$

Since $h_0(t_0,x_0) < \delta_0 < \gamma$, we have $h_0(t,x(t)) < \gamma$, $t \ge t_0$. In fact, if this is not true, there exists a $t_1 > t_0$, such that $h_0(t_1,x(t_1)) = \gamma$ and $h_0(t,x(t)) < \gamma$, $t \in [t_0,t_1)$. Thus,

$$D_-h_0(t_1,x(t_1)) = \liminf_{t\to t_1^-} \frac{1}{t-t_1}[h_0(t,x(t)) - h_0(t_1,x(t_1))] \ge 0.$$

This contradicts (iv).

Choose $T = T(\epsilon) = (a(\gamma)+1)/c(\delta))+\tau$. Then the system is (h_0,h)-eventually quasi-uniformly asymptotically stable with the choice of δ_0, τ_0, and T. To prove the theorem, it is sufficient to show that there exists a $t^* \in [t_0+\tau(\epsilon), t_0+T(\epsilon)]$, such that $h_0(t^*,x(t^*)) < \delta$. If this is not true, then $h_0(t,x(t)) \ge \delta$, $t \in [t_0+\tau(\epsilon), t_0+T(\epsilon)]$.

From (ii) and (iii), we have

$$0 \le V(t_0+T, x(t_0+T)) \le V(t_0+\tau, x(t_0+\tau)) - \int_{t_0+\tau}^{t_0+T} c(h_0(t,x(t)))dt$$

$$\le a(h_0(t_0+\tau, x(t_0+\tau))) - c(\delta)(T-\tau) < a(\gamma) - c(\delta)\frac{a(\gamma)+1}{c(\delta)} < 0.$$

The contradiction leads to the conclusion. ∎

The concept of Lyapunov function together with the theory of differential inequalities, provides a very general comparison principle under much less restrictive assumptions. If this sets up, Lyapunov function may be viewed as a transformation which reduces the study of a given complicated differential system to the study of relatively simpler scalar differential system.

Consider the differential system

$$x' = f(t,x), \quad x(t_0) = x_0 \tag{2.1}$$

and the scalar system

$$u' = g(t,u), \quad u(t_0) = u_0 \ge 0. \tag{2.2}$$

Theorem 3.3. *Assume that* (i) *and* (ii) *of Theorem* 3.1 *hold and there exists a function* $g \in C[R_+ \times R_+, R]$ *such that*
(iii) $D^+V(t,x) \le g(t, V(t,x))$ *for* $(t,x) \in S(h,\rho) \bigcap S^c(h_0, r)$, $t \ge \theta(r)$.
Then the eventually uniform stability of the set $u = 0$ *with respect to* (2.2) *implies that the system* (2.1) *is* (h_0, h)*-eventually uniformly stable.*

Proof. From (i), there exist a $\delta^* > 0$ and a function $\varphi \in K$, such that

$$h(t,x) \le \varphi(h_0(t,x)) \text{ provided that } h_0(t,x) < \delta^*.$$

Since $u = 0$ is eventually uniformly stable, for every $\epsilon \in (0,\rho)$, there exist two numbers $\overline{\delta} = \overline{\delta}(\epsilon)$, $\overline{\tau} = \overline{\tau}(\epsilon)$, such that $u(t,t_0,u_0) < b(\epsilon)$, $t \ge t_0 \ge \overline{\tau}$, provided that $u_0 \le \overline{\delta}$, where $u(t,t_0,u_0)$ is a solution of (2.2).

Choose $\delta \in (0,\delta^*]$ such that $a(\delta) < \overline{\delta}$ and $\varphi(\delta) < \epsilon$. Let $\tau = \max\{\overline{\tau}, \theta(\delta)\}$, then $h_0(t_0,x_0) < \delta$ implies $h(t_0,x_0) \le \varphi(h_0(t_0,x_0)) \le \varphi(\delta) < \epsilon$. The conclusion of the theorem holds with the choice of δ, τ. In fact, if this is not true, there exist $t_2 > t_1 > t_0$, such that

$$h_0(t_1,x(t_1)) = \delta, \quad h(t_2,x(t_2)) = \epsilon, \quad \text{and } (t,x) \in S(h,\rho) \bigcap S^c(h_0,\delta), \ t \in [t_1,t_2].$$

From (iii), we get

$$V(t,x(t)) \le r(t,t_1,u_0) \text{ for } t \in [t_1,t_2],$$

where $r(t,t_1,u_0)$ is a maximal solution of (2.2) with the initial value u_0 at time t_1 and $u_0 = V(t_1,x(t_1)) \le a(h_0(t_1,x(t_1))) = a(\delta) < \overline{\delta}$. Thus, we have

$$b(\epsilon) = b(h(t_2,x(t_2))) \le V(t_2,x(t_2)) \le r(t_2,t_1,u_0) < b(\epsilon).$$

This is a contradiction. The proof is complete. ■

Remark. By Theorem 3.3, we have the following conclusion: $g(t,u) = 0$ is admissible to yield that the system (2.1) is (h_0,h)-eventually uniformly stable.

Theorem 3.4. *Assume* (i)–(iii) *of Theorem* 3.3 *hold and*
(iv) h_0 *is locally lipschitzian in* x *and for every* $\gamma > 0$, *there exists a point* (t_0,x_0), *such that* $h_0(t_0,x_0) = \gamma$ *implies* $D_-h_0(t_0,x_0) < 0$.
Then the eventually uniform asymptotical stability of the set $u = 0$ *with respect to* (2.2) *implies that the system* (2.1) *is* (h_0,h)*-eventually uniformly asmptotically stable.*

Proof. By Theorem 3.3, the system (2.1) is (h_0,h)-eventually uniformly stable. So for $\rho > 0$, there exist $\delta_1 > 0$, $\tau_1 > \theta(\delta_1)$, such that

$$\text{if } h_0(t_0,x_0) < \delta_1, \text{ then } h(t,x(t)) < \rho, \ t \ge t_0 \ge \tau_1.$$

For every $\epsilon \in (0,\rho)$, there exist $\delta = \delta(\epsilon) > 0$, $\tau = \tau(\epsilon) \ge \theta(\delta)$, such that

$$\text{if } h_0(t_0,x_0) < \delta, \text{ then } h(t,x(t)) < \epsilon, \ t \ge t_0 \ge \tau. \tag{$*$}$$

Since $u = 0$ is eventually quasi-uniformly asymptotically stable, for $\epsilon \in (0,\rho)$, there exist $\delta_0 > 0$, $\tau_0 > 0$ and $T = T(\epsilon)$, such that $u(t, t_0, u_0) < b(\epsilon)$ for $t \geq t_0 + T$, $t_0 \geq \tau_0$, provided that $u_0 \leq \delta_0$.

Choose $\delta_2 > 0$, such that $a(\delta_2) \leq \delta_0$. Let $\overline{\delta} = \min\{\delta_1, \delta_2\}$, $\overline{\tau} = \max\{\tau_0, \tau_1\}$, $\overline{T} = \overline{T}(\epsilon) = T + \tau$, then with the choice of $\overline{\delta}$, $\overline{\tau}$, $\overline{T}$ the conclusion holds, namely, $h(t, x(t)) < \epsilon$, for $t \geq t_0 + \overline{T}$, $t_0 \geq \overline{\tau}$, provided that $h_0(t_0, x_0) < \overline{\delta}$.

If this is not true, there exists $t^* \geq t_0 + \overline{T}$, such that $h(t^*, x(t^*)) = \epsilon$. From $(*)$, there exists $\overline{t} \geq t_0 + \tau$, such that $h_0(t, x(t)) \geq \delta$ for $t \in [\overline{t}, t^*]$, thus $(t, x(t)) \in S(h,\rho) \bigcap S^c(h_0, \delta)$, $t \in [\overline{t}, t^*]$. Similar to the proof in Theorem 3.2, we can conclude that $h_0(t, x(t)) < \overline{\delta}$ provided that $h_0(t_0, x_0) < \overline{\delta}$ for $t \geq t_0$. From (iii), we have

$$V(t, x(t)) \leq r(t, \overline{t}, \overline{u}), \ t \in [\overline{t}, t^*],$$

where $r(t, \overline{t}, \overline{u})$ is a maximal solution of (2.2) with the intial value $\overline{u}$ at time $\overline{t}$, and $\overline{u} = V(\overline{t}, x(\overline{t})) \leq a(h_0(\overline{t}, x(\overline{t}))) < a(\delta) \leq \delta_0$.

From (ii), we have

$$b(\epsilon) = b(h(t^*, x(t^*))) \leq V(t^*, x(t^*)) \leq r(t^*, \overline{t}, \overline{u}) < b(\epsilon).$$

This is a contradiction. The proof is complete. ■

Remark. If $g(t, u) = -c(u)$, $c \in K$, then under the conditions of Theorem 3.4 the system (2.1) is (h_0, h)-eventually uniformly asymptotically stable.

Theorem 3.5. *Suppose that*

(i) *h_0 is uniformly finer than h;*

(ii) *there exists a function $V \in C[S(h,\rho), R_+]$, $V(t, x)$ is lipschitzian in x for a constant $L > 0$ and*

$$b(h(t,x)) \leq V(t,x) \leq a(h_0(t,x)) \ for \ (t,x) \in S(h,\rho) \bigcap S^c(h_0, r) \ and \ t \geq \theta(r),$$

where a, $b \in K$ and $\theta(r)$ is continuous and monotonic decreasing in r for $0 < r < \rho$;

(iii) *$D^+V(t,x)|_{(2.1)} \leq 0$ for $(t,x) \in S(h,\rho) \bigcap S^c(h_0, r)$ and $t \geq \theta(r)$.*

Then the perturbed system

$$x' = f(t,x) + R(t,x), \qquad x(t_0) = x_0 \tag{3.1}$$

is (h_0, h)-eventually uniformly stable, where $R \in C[S(h,\rho), R^n]$, and for every continuous function $x(t)$ such that $h(t, x(t)) \leq \rho^* < \rho$, $t \geq 0$, $\int_0^\infty \|R(s, x(s))\| ds < \infty$.

Proof. From (i), there exist a $\delta^* > 0$ and a function $\varphi \in K$, such that

$$h(t,x) \leq \varphi(h_0(t,x)) \text{ provided that } h_0(t,x) < \delta^*.$$

For every $\epsilon \in (0, \rho^*)$, we choose a $\delta = \delta(\epsilon) > 0$ such that $\varphi(\delta) < \epsilon$ and $2a(\delta) < b(\epsilon)$. Let $\tau_1(\epsilon) = \theta(\delta(\epsilon))$. Then since $\int_0^\infty \|R(s, x(s))\| ds < \infty$ for $h(t, x(t)) \leq \rho^* < \rho$, there exists a $\tau_2(\epsilon) > 0$, such that if $t_0 \geq \tau_2(\epsilon)$, then we have

$$\int_{t_0}^{\infty} \|R(s, x(s))\| ds < \frac{a(\delta)}{L}.$$

Let $\tau = \tau(\epsilon) = \max\{\tau_1(\epsilon), \tau_2(\epsilon)\}$. Then the system (3.1) is (h_0, h)-eventually uniformly stable with the choice of δ and τ. Observe that $h_0(t_0, x_0) < \delta$ implies $h(t_0, x_0) \le \varphi(h_0(t_0, x_0)) < \varphi(\delta) < \epsilon$. If this is not true, there exist $t_2 > t_1 > t_0$, such that

$$h_0(t_1, x(t_1)) = \delta, \qquad h(t_2, x(t_2)) = \epsilon$$

and

$$(t, x) \in S(h, \epsilon) \bigcap S^c(h_0, \delta), \quad t \in [t_1, t_2],$$

where $x(t) = x(t, t_0, x_0)$ is a solution of (3.1). From (iii), we have $D^+V(t, x)|_{(3.1)} \le L\|R(t, x)\|$ and consequently,

$$V(t_2, x(t_2)) \le V(t_1, x(t_1)) + L \int_{t_1}^{t_2} \|R(t, x(t))\| dt.$$

Hence

$$\begin{aligned} b(\epsilon) = b(h(t_2, x(t_2))) &\le V(t_2, x(t_2)) < V(t_1, x(t_1)) + L\frac{a(\delta)}{L} \\ &\le a(h_0(t_1, x(t_1))) + a(\delta) \le 2a(\delta). \end{aligned}$$

This is a contradiction. The proof is complete. ■

4. Applications

Example 1. Consider the system

$$\begin{cases} x' = -y + (1 - x^2 - y^2)x \exp(t - \dfrac{1}{1 - x^2 - y^2}), \\ y' = x + (1 - x^2 - y^2)y \exp(t - \dfrac{1}{1 - x^2 - y^2}). \end{cases} \tag{4.1}$$

Let $V(x, y) = (x^2 + y^2 - 1)^2$, $\quad h_0(x, y) = h(x, y) = |x^2 + y^2 - 1|$. Then

$$h^2(x, y) \le V(x, y) \le h_0^2(x, y) \quad \text{for } (x, y) \in R^2$$

and

$$\begin{aligned} D^+V(x, y)|_{(4.1)} =& 4(x^2 + y^2 - 1)(xx' + yy') \\ =& 4((x^2 + y^2 - 1)\Big[x^2 \exp(t - \frac{1}{1 - x^2 - y^2}) \\ &+ y^2 \exp(t - \frac{1}{1 - x^2 - y^2})\Big](1 - x^2 - y^2) \\ =& -4(x^2 + y^2 - 1)^2(x^2 + y^2) \exp(t - \frac{1}{1 - x^2 - y^2}) \le 0 \end{aligned}$$

for $(t, x, y) \in R \times R^2$. In paper [2], we can conclude that the system (4.1) is (h_0, h)-uniformly stable, but not (h_0, h)-uniformly asymptotically stable. However, according to Theorem 3.2, the system (4.1) is (h_0, h)-eventually uniformly asymptotically stable. In fact, if $h_0(x, y) = h(x, y) = |1 - x^2 - y^2| = 1 - x^2 - y^2$ for $0 < r \leq 1 - x^2 - y^2 < 2/3$, then

$$Dh_0(x, y) = -2xx' - 2yy' = -2(1 - x^2 - y^2)(x^2 + y^2) \exp(t - \frac{1}{1 - x^2 - y^2}) < 0,$$

$$D^+V(x, y) = -4(x^2 + y^2 - 1)^2(x^2 + y^2) \exp(t - \frac{1}{1 - x^2 - y^2}) \leq -4h_0^2(1 - h_0)$$

for $0 < r \leq 1 - x^2 - y^2 < 2/3$, $t \geq \theta(r) > 0$, where $\theta(r) = 1/r$. Since $V(x, y)$ and $h_0(x, y)$ satisfy the conditions of Theorem 3.2, the system is (h_0, h)-eventually uniformly asymptotically stable.

On the other hand, if $h_0(x, y) = h(x, y) = |x^2 + y^2 - 1| = x^2 + y^2 - 1$ for $0 < r \leq x^2 + y^2 - 1 < 1$, then

$$Dh_0(x, y) = 2xx' + 2yy' = 2(1 - x^2 - y^2)(x^2 + y^2) \exp(t - \frac{1}{1 - x^2 - y^2}) < 0,$$

$$D^+V(x, y) = -4(x^2 + y^2 - 1)^2(x^2 + y^2) \exp(t - \frac{1}{1 - x^2 - y^2}) \leq -4h_0^2(1 + h_0)$$

for $r \leq x^2 + y^2 - 1 < 1$, $t \geq \theta(r) = 1/r$. Since $V(x, y)$ and $h_0(x, y)$ satisfy the conditions of Theorem 3.2, then the system is (h_0, h)-eventually uniformly asymptotically stable.

Example 2. Consider the differential system

$$\begin{cases} x' = y + (1 - x^2 - y^2)x \exp(-t) \\ y' = -x + (1 - x^2 - y^2)y \sin^2 x \end{cases} \tag{4.2}$$

and the perturbed system

$$\begin{cases} x' = y + (1 - x^2 - y^2)x \exp(-t) + R_1(t, x, y) \\ y' = -x + (1 - x^2 - y^2)y \sin^2 x + R_2(t, x, y), \end{cases} \tag{4.3}$$

where $R_1(t, x, y) = R_2(t, x, y) = (x^2 + y^2 - 1)^2 t \exp(-t)$.

Let $V(x, y) = (x^2 + y^2 - 1)^2$, $h_0(x, y) = h(x, y) = |x^2 + y^2 - 1|$. Then we see that

$$h^2(x, y) \leq V(x, y) \leq h_0^2(x, y) \text{ for } (x, y) \in R^2,$$

$$\begin{aligned} D^+V(x, y)|_{(4.2)} =& 2(x^2 + y^2 - 1)(2xx' + 2yy') \\ =& 4(x^2 + y^2 - 1)(x^2 \exp(-t) + y^2 \sin^2 x)(1 - x^2 - y^2) \\ =& -4(x^2 + y^2 - 1)^2(x^2 \exp(-t) + y^2 \sin^2 x) \leq 0 \end{aligned}$$

and

$$\int_0^\infty \|R(s,x(s))\|ds = \int_0^\infty \sqrt{2}[x^2(t)+y^2(t)-1]^2 t\exp(-t)dt,$$

where $h(x(t),y(t)) = |x^2(t)+y^2(t)-1| \le \rho^*$. Hence, $\int_0^\infty \|R(s,x(s))\|ds < \infty$. By Theorem 3.5, the perturbed system (4.3) is (h_0,h)-eventually uniformly stable.

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