Vietnam Journal of Mathematics **28**:2(2000) 183–188

Vietnam Journal
of
MATHEMATICS

Short Communication

On Linear Regular Multipoint Boundary-Value Problems for Differential Algebraic Equations

Pham Ky Anh and Nguyen Van Nghi

Department of Mathematics, Vietnam National University
334 Nguyen Trai Street, Hanoi, Vietnam

Received August 20, 1999

1. Introduction

This paper deals with the so-called linear regular *multipoint boundary- value problems* (MPBVPs) for *differential-algebraic equations* (DAEs). It may be considered as a complement to works [1, 2], indicating further advantages of regular MPBVPs.

We begin by recalling some notations and concepts which will be frequently used in this article.

Let us consider the following linear MPBVP:

$$Lx := A(t)x' + B(t)x = q(t); \quad t \in J := [t_0,\ T], \tag{1}$$

$$\Gamma x := \int_{t_0}^{T} d\eta(t)x(t) = \gamma, \tag{2}$$

where $A, B \in C(J, \mathbb{R}^{n\times n})$ are contiuouns matrix-valued functions with $\det A(t) \equiv 0$, $\forall t \in J$; $\eta \in BV(J, \mathbb{R}^{n\times n})$ is a matrix-valued function of bounded variations, and $\gamma \in \mathbb{R}^n$ and $q \in C := C(J, \mathbb{R}^n)$ are given vector and vector-valued function, respectively.

In the remainder of the paper we assume that the pair of matrices $\{A, B\}$ satisfies the transferability condition [4], i.e.,

(i) There exists a continuously differentiable projector-function $Q \in C^1(J, \mathbb{R}^{n\times n})$, i.e., $Q^2(t) = Q(t)$ such that $\operatorname{Im} Q(t) = \operatorname{Ker} A(t)$ for all $t \in J$.

(ii) The matrix $G(t) := A(t) + B(t)Q(t)$ is non-singular for all $t \in J$.

Let $P(t) := I - Q(t)$, where I is the identity matrix, then $P \in C^1(J, \mathbb{R}^{n\times n})$ and $PQ = QP = 0$; $P^2 = P$.

Since

$$A(t)x' = A(t)\big[P(t)x(t)\big]' - A(t)P'(t)x(t), \tag{3}$$

we should restrict our consideration to the Banach space $\mathcal{X} := \{x \in C : Px \in C^1(J, \mathbb{R}^n)\}$, endowed with the norm $\|x\| := \|x\|_\infty + \|(Px)'\|_\infty$, where $\|x\|_\infty = \max\{\|x(t)\| : t \in J\}$ and $\|\cdot\|$ denotes a certain norm in $\mathbb{R}^n$.

From now on, we use the expression Ax' as an abbreviation of the right-hand side of (3).

Let $Y(t)$ be the fundamental solution matrix of the following *initial-value problem* (IVP): $Y' = (P'P_s - PG^{-1}B)Y$; $Y(t_0) = I$, where $Q_s := QG^{-1}B$ and $P_s := I - Q_s$ are canonical projections. It has been proved [4] that the matrix $X(t) := P_s(t)Y(t)P(t_0)$, with columns belonging to $\mathcal{X}$, satisfies the relations:

$$A(t)X' + B(t)X = 0; \; P(t_0)(X(t_0) - I) = 0.$$

Moreover, $\operatorname{Ker} X(t) = \operatorname{Ker} A(t)$ for every $t \in J$.

Now, define the so-called shooting matrix $D := \int_{t_0}^{T} d\eta(t)X(t)$ and a closed subspace $\operatorname{Im}\Gamma := \{\int_{t_0}^{T} d\eta(t)x(t) : x \in C\}$ of $\mathbb{R}^n$.

Definition 1. *The triplet $\{A, B, \eta\}$ is said to be regular if the shooting matrix D satisfies the conditions:*

$$\operatorname{Ker} D = \operatorname{Ker} A(t_0); \;\; \operatorname{Im} D = \operatorname{Im}\Gamma. \tag{4}$$

The regularity of $\{A, B, \eta\}$ guarantees the well-posedness of MPBVP (1)–(2). More precisely, what can be proved is the following:

Theorem 1. [1] *The regularity of the triplet $\{A, B, \eta\}$ is equivalent to the unique solvability of MPBVP (1)–(2) in the space $\mathcal{X}$ for all $q \in C$ and $\gamma \in \operatorname{Im}\Gamma$. Moreover, the solution of (1)–(2) depends continuously on the data (q, γ).*

From Theorem 1 and Banach's fixed point principle, one can derive an existence theorem for perturbed system:

$$Lx := A(t)x' + B(t)x = q(t) + \varepsilon f(x', x, t), \tag{5}$$

$$\Gamma x := \int_{t_0}^{T} d\eta(t)x(t) = \gamma + \varepsilon g(x). \tag{6}$$

Theorem 2. *Suppose that*

(i) *The triplet $\{A, B, \eta\}$ is regular.*

(ii) *$f : \mathbb{R}^n \times \mathbb{R}^n \times J \to \mathbb{R}^n$ is Lipschitz continuous in the first two variables, i.e.,*

$$\|f(\zeta, \varsigma, t) - f(\widetilde{\zeta}, \widetilde{\varsigma}, t)\| \le c_1\|\zeta - \widetilde{\zeta}\| + c_2\|\varsigma - \widetilde{\varsigma}\| \;\; \forall \zeta, \widetilde{\zeta}, \varsigma, \widetilde{\varsigma}, \in \mathbb{R}^n; \;\; \forall t \in J.$$

(iii) $\operatorname{Ker} A(t) \subset \operatorname{Ker} f'_\zeta(\zeta,\varsigma,t)\ \forall\zeta,\varsigma \in \mathbb{R}^n;\ \forall t \in J$. *Here, f is assumed to be continuously differentiable in the first variable and f'_ζ denotes the corresponding partial derivative.*

(iv) *The nonlinear boundary operator $g : C \to \operatorname{Im}\Gamma$ is Lipschitz continuous, i.e.,* $\|g(x) - g(y)\| \le c_3\|x - y\|_\infty;\ \forall x, y \in C$.

Then for a sufficiently small $\varepsilon > 0$, problem (5), (6) *is uniquely solvable for all $q \in C$ and $\gamma \in \operatorname{Im}\Gamma$. Moreover, the iterative process: $A(t)x'_{n+1} + B(t)x_{n+1} = q(t) + \varepsilon f(x'_n, x_n, t)$; $\int_{t_0}^T d\eta(t)x_{n+1}(t) = \gamma + \varepsilon g(x_n)$ is convergent at a geometrical rate.*

Using Theorems 1 and 2, we can prove the regularity of some perturbed systems.

Proposition 1. *The regularity of the triplet $\{A, B, \eta\}$ is stable under small pertubation of $B \in C(J, \mathbb{R}^{n\times n})$, i.e., if $\{A, B, \eta\}$ is regular, then, for any $C \in C(J, \mathbb{R}^{n\times n})$ and for a sufficiently small $\varepsilon > 0$, the triplet $\{A, B + \varepsilon C, \eta\}$ is also regular.*

Proposition 2. *Suppose that $\{A, B, \eta\}$ is a regular triplet. Then, for a sufficiently small $\varepsilon > 0$, the triplet $\{A + \varepsilon BP, B + \varepsilon BPP', \eta\}$ is also regular.*

We note that the pairs $\{A, B + \varepsilon C\}$ and $\{A + \varepsilon BP, B + \varepsilon BPP'\}$ in Propositions 1 and 2, respectively, are transferable.

Proposition 3. *Suppose that:*

(i) *$\{A, B, \eta\}$ is a regular triplet.*

(ii) *The perturbed boundary operator $\Gamma_\varepsilon(x) := \int_{t_0}^T d\eta_\varepsilon(t)x(t)$ with $\eta_\varepsilon \in BV(J, \mathbb{R}^{n\times n})$ satisfies two conditions:*

(a) $\operatorname{Im}\Gamma_\varepsilon \subset \operatorname{Im}\Gamma$.

(b) *The total variation $V_{t_0}^T(\eta - \eta_\varepsilon)$ is not greater than ε.*

Then, for a sufficiently small $\varepsilon > 0$, the triplet $\{A, B, \eta_\varepsilon\}$ is also regular.

From now on, let us consider a special MPBVP (1), (2) with $\eta(t) = \int_{t_0}^t C(s)ds$, where $C \in C(J,\ \mathbb{R}^{n\times n})$. Introducing a new variable $y(t) := \int_{t_0}^t C(s)x(s)ds$, we can reduce (1), (2) to a two-point BVP for an enlarged system:

$$\overline{A}(t)z' + \overline{B}(t)z = \overline{q}(t), \tag{7}$$

$$Mz(t_0) + Nz(T) = \overline{\gamma}, \tag{8}$$

where $\overline{A} := \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$; $\overline{B} := \begin{pmatrix} B & 0 \\ -C & 0 \end{pmatrix}$; $M := \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$; $N := \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$;

$\overline{\gamma} := \begin{pmatrix} \gamma \\ 0 \end{pmatrix}$; $\overline{q} := \begin{pmatrix} q \\ 0 \end{pmatrix}$; $z := \begin{pmatrix} x \\ y \end{pmatrix}$.

Obviously, boundary condition (8) can be rewritten as: $\overline{\Gamma}z := \int_{t_0}^{T} d\overline{\eta}z(t) = \overline{\gamma}$ with

$$\overline{\eta}(t) = \begin{cases} -M & t = t_0 \\ 0 & t_0 < t < T \\ N & t = T. \end{cases}$$

Defining the enlarged projections $\overline{Q} = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}$; $\overline{P} = \overline{I} - \overline{Q} := \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}$ and taking into account the fact that $\overline{G} = \overline{A} + \overline{BQ} := \begin{pmatrix} G & 0 \\ -CQ & I \end{pmatrix}$; $\overline{G}^{-1} := \begin{pmatrix} G^{-1} & 0 \\ CQG^{-1} & I \end{pmatrix}$, we come to the conclusion that the pair $\{\overline{A}, \overline{B}\}$ is also transferable.

However, if Im $\Gamma \neq \mathbb{R}^n$, then the triplet $\{\overline{A}, \overline{B}, \overline{\eta}\}$ is not regular with respect to the space $\overline{\mathcal{X}} = \{z = (x, y)^T \in C : Px, y \in C^1\}$.

Thus, it should be more rational to implement approximate methods directly to regular MPBVPs than to enlarged two-point BVPs.

Let $t_0 < t_1 < \cdots < t_m = T$ be a given partition of the interval $J = [t_0, T]$. Denote by $X_i(t)$ the fundamental matrix satisfying relations $A(t)X_i' + B(t)X_i = 0,\ t \in [t_i, t_{i+1}];\ P_i[X_i(t_i) - I] = 0\ (i = 0, ..., m-1)$, where $P_i := P(t_i)$.

We are looking for a solution of (1), (2) of the form

$$x(t) = x(t, s_0, s_1, ..., s_{m-1}) := x_i(t) \text{ if } t \in [t_i, t_{i+1}],\ (i = 0, ..., m-1),$$

where $x_i(t)$ is a solution of the IVP:

$$\begin{cases} A(t)x_i' + B(t)x_i = q(t), \\ P_i(x_i(t_i) - s_i) = 0. \end{cases} \tag{9}$$

The shooting vectors $s_i (i = 0, ..., m-1)$ are defined from the matching conditions:

$$P_i(x_{i-1}(t_i) - s_i) = 0 \tag{10}$$

and the boundary condition (2):

$$\int_{t_0}^{T} C(t)x(t)dt = \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} C(t)x_i(t)dt = \gamma. \tag{11}$$

According to [4], $x(t, s_0, ..., s_{m-1})$ is a solution of (1) and therefore, by virtue of (11), it is also a (unique) solution of (1), (2).

Denote by $Y_i(t)$ the fundamental matrix of the IVP: $Y_i' = (P'P_s - PG^{-1}B)Y_i;\ Y_i(t_i) = I;\ t \in [t_i, t_{i+1}]\ (i = 0, ..., m-1)$.

From (9), it follows that

$$x_i(t) = X_i(t)s_i + f_i(t)\ ;\ t \in [t_i, t_{i+1}]\ (i = 0, ..., m-1), \tag{12}$$

where

$$f_i(t) = X_i(t) \int_{t_i}^{t} Y_i^{-1}(\tau) P_s(\tau)(I + P'(\tau)) G^{-1}(\tau) q(\tau) d\tau + Q(t) G^{-1}(t) q(t).$$

The matching condition (10) implies that

$$P_i(X_{i-1}(t_i)s_{i-1} - s_i) = r_i, \tag{13}$$

where $r_i = -P_i f_{i-1}(t_i)$. Using (11), we have

$$\sum_{i=0}^{m-1} C_i s_i = \overline{\gamma}, \tag{14}$$

where $C_i = \int_{t_i}^{t_{i+1}} C(t) X_i(t) dt$ and $\overline{\gamma} = \gamma - \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} C(t) f_i(t) dt$. Let $Q_i := I - P_i$.

Theorem 3. *Let $\{A, B, \eta\}$ be a regular triplet. Then the shooting vectors $\{s_i\}_{i=0}^{m-1}$ satisfying conditions* (13), (14) *and additional relations $Q_i s_i = 0$ ($i = 0, ..., m-1$) are uniquely determined. Moreover, the solution of MPBVP* (1), (2) *is given by* (12).

Finally, let us consider a MPBVP for a weakly non-linear DAE:

$$Lx := A(t)x' + B(t)x = q(t) + \varepsilon f(x, t), \tag{15}$$

$$\Gamma x := \int_{t_0}^{T} C(t) x(t) dt = \gamma. \tag{16}$$

Suppose that the triplet $\{A, B, \eta\}$ is regular and the non-linear part $f(x, t)$ is Lipschitz continuous in x. From Theorem 2, it follows that for a sufficiently small $\varepsilon > 0$, MPBVP (15), (16) has a unique solution and the Picard method applied to (15), (16) is convergent. However, avoiding the stiffness effect of the left-hand side of (15), we should implement the multiple iteration method (see [3]) for solving (15), (16).

Suppose that the $(j-1)$th iteration $x^{j-1}(t)$ has been found. Then the next approximation $x^j(t)$ can be defined as

$$x^j(t) = x_i^j(t), \quad t \in [t_i, t_{i+1}] \ (i = 0, ..., m-1),$$

where $x^0(t)$ is a solution of linear MPBVP (1), (2) obtained by the above-mentioned multishooting method, and $A(t)(x_i^j)' + B(t)x_i^j = q(t) + \varepsilon f(x^{j-1}, t)$; $P_i(x_i^j(t_i) - s_i^j) = 0$ ($j \geq 1$). The matching conditions give $P_i(x_{i-1}^j(t_i) - s_i^j) = 0$. Further, from boundary condition (16) we get $\sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} C(t) x_i^j(t) dt = \gamma$. Thus, we define

$$x_i^j(t) = X_i(t) s_i^j + f_i^j(t), \quad t \in [t_i, t_{i+1}], \tag{17}$$

where

$$f_i^j(t) = X_i(t)\int_{t_i}^{t} Y_i^{-1}(\tau)P_s(\tau)(I+P'(\tau))G^{-1}(\tau)\big[q(\tau)+\varepsilon f(x^{j-1},\tau)\big]d\tau$$
$$+Q(t)G^{-1}(t)\big[q(t)+\varepsilon f(x^{j-1},t)\big].$$

Putting

$$r_i^j = -P_i f_{i-1}^j(t_i);\ \ \gamma^j = \gamma - \sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} C(t)f_i^j(t)dt,$$

we come to a system, similar to those of (13), (14):

$$\sum_{i=0}^{m-1} C_i s_i^j = \gamma^j;\ \ P_i X_{i-1}(t_i)s_{i-1}^j - P_i s_i^j = r_i^j\ \ (i=1,...,m-1).$$

Adding to this system m relations $Q_i s_i^j = 0\ \ (i=0,...,m-1)$, we can define the shooting vectors s_i^j and then determine $x_i^j(t)\ (i=0,...,m-1)$ by (17).

Theorem 4. *Suppose that the triplet $\{A, B, \eta\}$ is regular and $f(x,t)$ is Lipschitz continuous in x. Then, for a sufficiently small $\varepsilon > 0$, the multiple iteration method is convergent at a geometrical rate.*

Detailed proofs and numerical experiments will be given in a forthcoming paper.

References

1. P. K. Anh, Multipoint boundary-value problems for transferable differential-algebraic equations. I-Linear case, *Vietnam J. Math.* **25** (1997) 347–358.
2. P. K. Anh, Multipoint boundary-value problems for transferable differential-algebraic equations. II-Quasilinear case, *Vietnam J. Math.* **26** (1998) 337–349.
3. N. T. Bang, A combination of Picard's method and time-decomposition technique for the stiff nonlinear two-point boundary-value problems, *Control and Cybernetics* **11** (3-4) (1982) 93–108.
4. E. Griepentrog and R. März, *Differential-Algebraic Equations and Their Numerical Treatment*, Teubner-Text Math., Vol. 88, Teubner, Leipzig, 1986.