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Some Common Fixed Point Theorems for Mappings in Metric and Menger Spaces*

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Abstract. We prove generalizations of results of C. S. Wong, T.-H. Chang and Do Hong Tan with applications to Menger probabilistic metric spaces and random operator equations.

1. Introduction and Preliminaries

Let f_1, f_2 be two self-mappings of a complete metric space (X, d). We denote

$$m_1(f_1x, f_2y) = \max\Big\{d(x,y),\ \frac{d(x,f_1x)+d(y,f_2y)}{2},\ \frac{d(x,f_2y)+d(y,f_1x)}{2}\Big\},$$

$$m_2(f_1x, f_2y) = \max\Big\{d(x,y),\ d(x,f_1x),\ d(y,f_2y),\ \frac{d(x,f_2y)+d(y,f_1x)}{2}\Big\}.$$

In what follows by $m(f_1x, f_2y)$ we mean either of them. We shall be concerned with the following condition: For each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\varepsilon \leq m(f_1x, f_2y) < \varepsilon + \delta \text{ for } x \neq y \text{ implies } d(f_1x, f_2y) < \varepsilon. \qquad (1.1)$$

Note that the condition of type (1.1) not necessary for distinct x, y was considered in [3] which generalized the concept of (ε, δ)-contractive mappings [4]. Under this weaker condition (1.1) we shall prove some common fixed point theorems for the pair f_1, f_2. Along these lines the next theorem generalizing the

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result of [10] is that if f_1, f_2 satisfy the so-called g-generalized contractive condition

$$d(f_1x, f_2y) \le g(m(f_1x, f_2y)) \quad \forall x \neq y \in X, \tag{1.2}$$

where g is a self-function of $\mathbb{R}^+$ satisfying the following property:

(G) g is upper-semicontinuous, $g(0) = 0$ and $g(t) < t \ \forall t > 0$,

then one of f_1, f_2 has a fixed point. Moreover, if both f_1, f_2 have fixed points, then f_1, f_2 have a unique common fixed point in X, which is also a unique fixed point for each f_1, f_2. There are simple examples (cf. [9]) showing that the conclusion of the theorems above is the best possible.

Before proceeding to the case of probabilistic (random) metric spaces, let us mention some definitions [5–7]. Let Δ_0 denote the set of all distribution functions F with $F(0) = 0$ (F is non-decreasing, left-continuous and $\sup_{t\in\mathbb{R}} F(t) = 1$). A probabilistic metric space (a PM-space) is an ordered pair $(X, \mathcal{F})$ consisting of a non-empty set X and a symmetric mapping $\mathcal{F} : X \times X \to \Delta_0$ ($\mathcal{F}(x, y)$ is denoted by $F_{x,y}$ for $x, y \in X$) which satisfies the following conditions:

(1) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$.

(2) If $F_{x,z}(t) = 1$ and $F_{z,y}(s) = 1$, then $F_{x,y}(t + s) = 1$ for all $x, y, z \in X$ and $t, s > 0$.

A Menger space is a triplet $(X, \mathcal{F}, T)$, where $(X, \mathcal{F})$ is a PM-space, T is a triangular norm (t-norm), and the Menger triangular inequality

$$F_{x,y}(t + s) \ge T(F_{x,z}(t), F_{z,y}(s))$$

holds for all $x, y, z \in X$ and $t, s > 0$. Recall that a t-norm T is a commutative, associative, and non-decreasing mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ such that $T(0, 0) = 0$, $T(a, 1) = a$. A t-norm T_1 is stronger than a t-norm T_2 (written as $T_1 \ge T_2$) if $T_1(a, b) \ge T_2(a, b)$, $\forall a, b \in [0, 1]$. If, in addition, there is at least one pair (a, b) with strict inequality, then we say T_1 strictly stronger than T_2. There are two important t-norms: $T(a, b) := \min\{a, b\}$ and $T_m(a, b) := \max\{a+b-1, 0\}$ which will be used frequently in the sequel. The case $(X, \mathcal{F}, \min)$ was studied extensively (see, e.g. [1, 3] and the cited references therein). In this case, for each $\lambda \in (0, 1)$, one can define a pseudo-metric d_λ by putting $d_\lambda(x, y) = \sup\{t : F_{x,y}(t) \le 1 - \lambda\}$. In fact according to [7, 8] one can deal with the following three equivalent objects:

(1) $(X, \mathcal{F}, \min)$ is a Menger space.

(2) $(X, \mathcal{F})$ is pseudo-metrically generated by the family $\mathcal{D} := \{d_\lambda\}$ endowed with a natural measure – the Lebesgue measure μ for $\Omega := (0, 1)$ where $\mathcal{D}$ is linearly ordered by the relation $d_{\lambda_1} \le d_{\lambda_2}$ if and only if $\lambda_1 \ge \lambda_2$.

(3) $(X, \mathcal{F})$ is isometric to an E-space consisting of functions from $(\Omega, \mathcal{B}, \mu)$ into the metric space (M, δ) such that, for each $\lambda \in \Omega : d_\lambda(x, y) = \delta(x(\lambda), y(\lambda))$, where M is the set of equivalence classes of the explicitly pseudo-metrizable space $X \times \Omega$.

Our next aim is to extend the method here to the class of Menger spaces with t-norm $T \ge T_m$, and since by [7] every E-space is a Menger space w.r.t. t-norm T_m, we can apply the results of this type to the theory of random operator equations.

2. Common Fixed Point Theorems

Theorem 2.1. *Let (X,d) be a complete metric space, and f_1, f_2 two mappings of X into itself. Assume that condition (1.1) holds with $m = m_1$. Then at least one of f_1, f_2 has a fixed point. If both f_1, f_2 have fixed points, then they have a unique common fixed point which is also a unique fixed point for each of them. In particular, it is so if the implication of type (1.1) holds for all x, y in X.*

Proof. Let $x_0 \in X$ be arbitrary, and we construct the sequence $\{x_n\}$ as follows: $x_{2n+1} := f_1 x_{2n}$, $x_{2n+2} := f_2 x_{2n+1}$. One may assume that $x_n \neq x_{n+1}, \forall n$, otherwise some of them is a fixed point of f_1 or f_2. Putting $d_n := d(x_n, x_{n+1})$, as in the proof of Theorem 1 in [3], one sees that $\{d_n\}$ converges to 0 (at this step the condition (1.1) is sufficient for that purpose).

Following the methods of Meir–Keeler and Wong, we now prove that $\{x_n\}$ is a Cauchy sequence contrariwise. Assume that there exists an $\varepsilon > 0$ such that $\forall k \in \mathbf{N}$, $\exists n > m > k$ such that $d(x_n, x_m) \geq 2\varepsilon$. From (1.1) we choose $\delta > 0$ for this ε, and put $\alpha = \min\{\varepsilon, \delta\}$. The argument similar to that of [3] shows that one can consider sufficiently large k so that $d_i < \alpha/4$, $\forall i \geq k$, and for each such k, $\exists p(k) > q(k) > k$ such that

$$\varepsilon + \frac{\alpha}{4} \leq a_k < \varepsilon + \frac{\alpha}{2}, \tag{2.1}$$

where $a_k := d(x_{p(k)}, x_{q(k)})$ and

$$d(x_i, x_{q(k)}) \leq d(x_{i+1}, x_{q(k)}) + d_i,\ \ d(x_{i+1}, x_{q(k)}) \leq d(x_i, x_{q(k)}) + d_i, \tag{2.2}$$

for each $i \in \{q(k), \ldots, p(k)\}$. Hence,

$$d(x_{p(k)-1}, x_{q(k)}) < \varepsilon + \frac{\alpha}{4}. \tag{2.3}$$

Since in view of (2.3) and the triangle inequality

$$\varepsilon + \frac{\alpha}{4} \leq a_k \leq d_{p(k)-1} + d(x_{p(k)-1}, x_{q(k)}) < d_{p(k)-1} + \varepsilon + \frac{\alpha}{4},$$

one sees that $\{a_k\}$ converges to $\varepsilon + \alpha/4$ from the right. Let

$$I_1 := \{k:\ p(k) \text{ even},\ q(k) \text{ odd}\},$$

$$I_2 := \{k:\ p(k) \text{ odd},\ q(k) \text{ odd}\},$$

$$I_3 := \{k:\ p(k) \text{ odd},\ q(k) \text{ even}\},$$

$$I_4 := \{k:\ p(k) \text{ even},\ q(k) \text{ even}\}.$$

Then at least one of I_i, $i = 1, \cdots, 4$ is infinite. If I_1 is infinite, since

$$\varepsilon + \frac{\alpha}{4} \leq a_k \leq d_{p(k)-1} + d(x_{p(k)-1}, x_{q(k)-1}) + d_{q(k)-1},$$

so $\{d(x_{p(k)-1}, x_{q(k)-1})\}$ converges to $\varepsilon + \alpha/4$. Hence, there is at least a $k \in I_1$ such that $x_{p(k)-1} \neq x_{q(k)-1}$. For this k, we get $m_1(f_1 x_{q(k)-1}, f_2 x_{p(k)-1}) < \varepsilon + \delta$. Then in view of (1.1) one obtains $d(x_{q(k)}, x_{p(k)}) < \varepsilon$, a contradiction to (2.1). Now, suppose that I_2 is infinite. Since

$$\varepsilon + \frac{\alpha}{4} \leq a_k \leq d_{p(k)-2} + d_{p(k)-1} + d(x_{p(k)-2}, x_{q(k)-1}) + d_{q(k)-1},$$

so $\{d(x_{p(k)-2}, x_{q(k)-1})\}$ converges to $\varepsilon + \alpha/4$. Hence, there is at least a $k \in I_2$ such that $x_{p(k)-2} \neq x_{q(k)-1}$. For this k we get $m_1(f_1 x_{q(k)-1}, f_2 x_{p(k)-2}) < \varepsilon + \delta$. Then in view of (1.1) one obtains $d(x_{q(k)}, x_{p(k)-1}) < \varepsilon$. On the other hand, by (2.2) and (2.1): $d(x_{q(k)}, x_{p(k)-1}) \geq d(x_{q(k)}, x_{p(k)}) - d_{p(k)-1} \geq \varepsilon + \frac{\alpha}{4} - \frac{\alpha}{4} = \varepsilon$, a contradiction. Similarly for the other two cases except the roles of f_1, f_2 interchange. Thus, the sequence $\{x_n\}$ is convergent, say to a limit $x \in X$. Since $d_n > 0$, $\forall n$, one of $I := \{n : x \neq x_{2n+1}\}$, $J : \{n : x \neq x_{2n}\}$ is infinite. If I is infinite, then assume $\varepsilon := d(x, f_1 x) > 0$. For the value $\varepsilon/2$, we choose δ such that (1.1) holds. We have $d(x, f_1 x) \leq d(x, x_{2n+2}) + d(f_2 x_{2n+1}, f_1 x)$ and

$$m_1(f_1 x, f_2 x_{2n+1}) = \max\Big\{d(x, x_{2n+1}), \frac{1}{2}(d(x, f_1 x) + d(x_{2n+1}, x_{2n+2})), \frac{1}{2}(d(x, x_{2n+2}) + d(x_{2n+1}, f_1 x))\Big\}.$$

So $m_1(f_1 x, f_2 x_{2n+1}) < \varepsilon/2 + \delta$ for n sufficiently large. Hence, by (1.1), one gets $d(x, f_1 x) < \varepsilon$, a contradiction. Thus, $x = f_1 x$. Similarly, for the case when J is infinite, $x = f_2 x$. ■

One can have an easy application of the above result to Menger spaces with $T = \min$. Recall that the (ε, λ)-topology in a Menger space $(X, \mathcal{F}, T)$ can be defined by the family $\{U_x(\varepsilon, \lambda);\ x \in X,\ \varepsilon > 0,\ \lambda \in (0,1)\}$ of (ε, λ)-neighborhoods, where

$$U_x(\varepsilon, \lambda) := \{y \in X;\ \ F_{x,y}(\varepsilon) > 1 - \lambda\}.$$

If $\sup_{a \in (0,1)} T(a, a) = 1$, then $(X, \mathcal{F}, T)$ is a Hausdorff topological space in the (ε, λ)-topology. It is easy to see that $d_\lambda(x, y) = \inf\{\varepsilon > 0 : y \in U_x(\varepsilon, \lambda)\}$ in the case $T = \min$. The family $\{d_\lambda\}$ generates the same topology in $(X, \mathcal{F}, \min)$. In particular, it satisfies the following property: $d_\lambda(x, y) = 0$, $\forall \lambda \in (0, 1)$ if and only if $x = y$. As an immediate consequence of Theorem 2.1 of this paper and Theorem 3 of [3], one obtains

Corollary 2.2. *Let $(X, \mathcal{F}, \min)$ be a complete Menger space, and f_1, f_2 two mappings of X into itself. Assume, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all $x \neq y$ in X*

$$F_{f_1 x, f_2 y}(\varepsilon) \geq \min\Big\{F_{x,y}(\varepsilon + \delta), \max(F_{x, f_1 x}(\varepsilon + \delta), F_{y, f_2 y}(\varepsilon + \delta)), \max(F_{x, f_2 y}(\varepsilon + \delta), F_{y, f_1 x}(\varepsilon + \delta))\Big\}. \qquad (2.4)$$

Then at least one of f_1, f_2 has a fixed point. If both f_1, f_2 have fixed points, then they have a unique common fixed point which is also a unique fixed point for each of them. In particular, it is so if (2.4) *holds for all x, y in X.*

Theorem 2.3. *Let (X, d) be a complete metric space, and f_1, f_2 two mappings of X into itself such that one of them is continuous. Assume that condition* (1.1) *holds with $m = m_2$. Then at least one of f_1, f_2 has a fixed point. If both f_1, f_2 have fixed points, then they have a unique common fixed point which is also a unique fixed point for each of them. In particular, it is so if the implication of type* (1.1) *holds for all x, y in X.*

Proof. As in the proof of Theorem 2.1 the sequence $\{x_n\}$ is convergent to a limit $x \in X$. If f_1 is continuous, then $x = f_1 x$.

Remark 1. The example $X = \{x_n = 2^{-n},\ n = 0, 1, 2, ...,\ x_\infty = 0\}$ and $f_1 = f_2$, $f_1(x_n) := x_{n+1}$, $n = 0, 1, 2, ...$, $f_1(x_\infty) = x_0$ in [3] shows that the continuity assumption in Theorem 2.3 above is essential. Since in Theorem 2.1 the continuity is not assumed, one easily checks that f_1, f_2 do not satisfy (1.1) with $m = m_1$, and hence have no fixed points.

Corollary 2.4. *Let $(X, \mathcal{F}, min)$ be a complete Menger space, and f_1, f_2 two mappings of X into itself with f_1 or f_2 continuous. Assume, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all $x \neq y$ in X,*

$$F_{f_1x, f_2y}(\varepsilon) \geq \min\Big\{F_{x,y}(\varepsilon + \delta), F_{x,f_1x}(\varepsilon + \delta), F_{y,f_2y}(\varepsilon + \delta), \max\Big(F_{x,f_2y}(\varepsilon + \delta), F_{y,f_1x}(\varepsilon + \delta)\Big)\Big\}. \tag{2.5}$$

Then at least one of f_1, f_2 has a fixed point. If both f_1, f_2 have fixed points, then they have a unique common fixed point which is also a unique fixed point for each of them. In particular, it is so if (2.5) *holds for all x, y in X.*

Theorem 2.5. *Let (X, d) be a complete metric space, and f_1, f_2 two mappings of X. Assume that condition* (1.2) *holds. Then at least one of f_1, f_2 has a fixed point. If both f_1, f_2 have fixed points, then they have a unique common fixed point which is also a unique fixed point for each of them. In particular, it is so if condition* (1.2) *holds for all x, y in X.*

Proof. Obviously one can assume $m = m_2$. As was remarked in [3] condition (1.2) implies condition (1.1). So one constructs the sequence $\{x_n\}$ as before which is convergent to a limit $x \in X$.

Since $d_n > 0$, $\forall n$, one of $I := \{n : x \neq x_{2n+1}\}$, $J := \{n : x \neq x_{2n}\}$ is infinite. If I is infinite, then assume $t_0 := d(x, f_1 x) > 0$. We have $d(x, f_1 x) \leq d(x, x_{2n+2}) + d(f_2 x_{2n+1}, f_1 x)$ and

$$\begin{aligned} d(f_1x, f_2x_{2n+1}) &\le g(m(f_1x, f_2x_{2n+1})) \\ &= g\Big(\max\Big\{d(x, x_{2n+1}), d(x, f_1x), d(x_{2n+1}, x_{2n+2}), \\ &\qquad \frac{1}{2}(d(x, x_{2n+2}) + d(x_{2n+1}, f_1x))\Big\}\Big). \end{aligned}$$

So by the upper semicontinuity of g and $g(t_0) < t_0$, one can choose $\varepsilon = t_0 - g(t_0) - \varepsilon_0 > 0$ such that, for this ε : $g(m(f_1x, f_2x_{2n+1})) < g(t_0) + \varepsilon = t_0 - \varepsilon_0$ for n sufficiently large. Hence, we get

$$t_0 = d(x, f_1x) < \varepsilon_0 + t_0 - \varepsilon_0 = t_0,$$

a contradiction. Thus, $x = f_1x$. Similarly, for the case when J is infinite, $x = f_2x$.

Remark 2. The example with $X = \{1,2\}$ and f_1, f_2 two different mappings onto [9] shows that the conclusion of the theorems above is the best possible.

Corollary 2.6. *Let $(X, \mathcal{F}, \min)$ be a complete Menger space, f_1, f_2 two mappings of X into itself. Assume that there exists a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying condition (G) such that, for all $x \neq y$ in X and $t > 0$,*

$$F_{f_1x,f_2y}(g(t)) \ge \min\{F_{x,y}(t), F_{x,f_1x}(t), F_{y,f_2y}(t), \max(F_{x,f_2y}(t), F_{y,f_1x}(t))\}. \tag{2.6}$$

Then at least one of f_1, f_2 has a fixed point. If both f_1, f_2 have fixed points, then they have a unique common fixed point which is also a unique fixed point for each of them. In particular, it is so if (2.6) holds for all x, y in X.

3. Applications to Menger Spaces with $T \ge T_m$

Let $(X, \mathcal{F}, T)$ be a Menger space. It is well known that if t-norm T satisfies $\sup_{a\in(0,1)} T(a,a) = 1$, then in the (ε, λ)-topology, X is a metrizable topological space. In the case $T \ge T_m$, there is a metric with nice properties, namely

$$\beta(x, y) := \inf\{u : F_{x,y}(u^+) > 1 - u\}.$$

Indeed, by the definition of β, the only property of metrics one has to verify is the triangle inequality. We shall prove it on the contrary: Assume there are x, y, z in X such that $\beta(x, y) > \beta(x, z) + \beta(z, y)$. Choose $0 < \varepsilon := \beta(x, y) - \beta(x, z) - \beta(z, y)$ so that

$$\beta(x, y) > t_1 + t_2, \tag{3.1}$$

where $t_1 := \beta(x, z) + \varepsilon/3$, $t_2 := \beta(z, y) + \varepsilon/3$. In view of the definition of β,

$$\begin{aligned} &F_{x,y}(\beta(x, y)) \le 1 - \beta(x, y), \\ &F_{x,z}(t_1) > 1 - t_1, \quad F_{z,y}(t_2) > 1 - t_2. \end{aligned}$$

On the other hand, by using properties of t-norms (taking into account $T \geq T_m$) and distribution functions, we have

$$\begin{aligned}1-\beta(x,y) &\geq F_{x,y}(\beta(x,y)) \geq F_{x,y}(t_1+t_2)\\ &\geq T(F_{x,z}(t_1), F_{z,y}(t_2))\\ &\geq T(1-t_1, 1-t_2)\\ &\geq T_m(1-t_1, 1-t_2)\\ &= (1-t_1)+(1-t_2)-1 = 1-t_1-t_2,\end{aligned}$$

a contradiction to (3.1).

Theorem 3.1. *Let $(X, \mathcal{F}, T)$ be a complete Menger space with $T \geq T_m$, and f_1, f_2 two mappings of X into itself. Assume that there exists a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying condition* (G) *such that, for all $x \neq y$ in X and $t > 0$,*

$$1 - F_{f_1x,f_2y}(g(t)) \leq g\Big(1 - \min\Big\{F_{x,y}(t), F_{x,f_1x}(t), F_{y,f_2y}(t), [F_{x,f_2y}(t) + F_{y,f_1x}(t)]/2\Big\}\Big). \tag{3.2}$$

Then at least one of f_1, f_2 has a fixed point. If both f_1, f_2 have fixed points, then they have a unique common fixed point which is also a unique fixed point for each of them. In particular, it is so if (3.2) *holds for all x, y in X.*

Proof. By Proposition 1 of [1], there exists a continuous and strictly increasing (hence invertible) self-function f of $\mathbb{R}^+$ such that $g(t) \leq f(t) < t$, $\forall\, t > 0$. We now show that condition (1.2) of Theorem 2.5 holds w.r.t. the metric β. Assume the contrary that there exist $x \neq y$ in X such that

$$\beta(f_1x, f_2y) > f(m(f_1x, f_2y)),$$

i.e.,

$$t := f^{-1}(\beta(f_1x, f_2y)) > m(f_1x, f_2y).$$

So in view of the properties of the metric β and by using the monotony of f and distribution functions, we have

$$\begin{aligned}&1-F_{f_1x,f_2y}(g(t)) \geq 1-F_{f_1x,f_2y}(f(t)) \geq \beta(f_1x,f_2y) > f(m(f_1x,f_2y))\\ &\geq f(\max\{1-F_{x,y}(\beta(x,y)^+), 1-F_{x,f_1x}(\beta(x,f_1x)^+), 1-F_{y,f_2y}(\beta(y,f_2y)^+),\\ &\qquad 1-[F_{x,f_2y}(\beta(x,f_2y)^+)+F_{y,f_1x}(\beta(y,f_1x)^+)]/2\})\\ &\geq f(\max\{1-F_{x,y}(t), 1-F_{x,f_1x}(t), 1-F_{y,f_2y}(t), 1-[F_{x,f_2y}(t)+F_{y,f_1x}(t)]/2\})\\ &= f(1-\min\{F_{x,y}(t), F_{x,f_1x}(t), F_{y,f_2y}(t), [F_{x,f_2y}(t)+F_{y,f_1x}(t)]/2\})\\ &\geq g(1-\min\{F_{x,y}(t), F_{x,f_1x}(t), F_{y,f_2y}(t), [F_{x,f_2y}(t)+F_{y,f_1x}(t)]/2\}),\end{aligned}$$

a contradiction to (3.2).

Corollary 3.2. *Let $(X, \mathcal{F}, T)$ be a complete Menger space with $T \geq T_m$, and f_1, f_2, h_1, h_2 four mappings of X into itself such that*

(a) $f_1(X) \subset h_2(X)$, $f_2(X) \subset h_1(X)$,
(b) f_1, h_1 *are probabilistic compatible, and so are* f_2, h_2,
(c) *one of the mappings is continuous,*
(d) *there exists a function* $g : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfying condition* (G) *such that for all* x, y *in* X *and* $t > 0$,

$$1 - F_{f_1x,f_2y}(g(t)) \leq g(1 - \min\{F_{h_1x,h_2y}(t), F_{h_1x,f_1x}(t), F_{h_2y,f_2y}(t), [F_{h_1x,f_2y}(t) + F_{h_2y,f_1x}(t)]/2\}). \tag{3.3}$$

Then four mappings have a unique common fixed point.

Recall that two self-mappings f, h of a PM-space $(X, \mathcal{F})$ are said to be probabilistic compatible if $\lim_{n\to\infty} F_{fhx_n,hfx_n}(t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} F_{fx_n,hx_n}(t) = 1$ for all $t > 0$. In particular, $fhx = hfx$ if $fx = hx$ (by taking $x_n = x$, $\forall n$).

Proof of the Corollary. Let

$$m_h(f_1x, f_2y) = \max\left\{ d(h_1x, h_2y),\ d(h_1x, f_1x),\ d(h_2y, f_2y),\ \frac{d(h_1x, f_2y) + d(h_2y, f_1x)}{2} \right\}.$$

In view of Theorem 2 of [3], one has to verify the metric condition for the case of four mappings:

$$\beta(f_1x, f_2y) \leq f(m_h(f_1x, f_2y))$$

for all x, y in X. Since (3.3) is a version of (3.2) with h_1, h_2 involved, the proof is similar to that of Theorem 3.1 and can be omitted.

Remark 3. Recently, Cho, Ha and Chang ([2, Theorem 3.2] have proved the same conclusion of Corollary 3.2, but under stronger conditions and by a different method.

We now apply the results above in showing the existence of a unique solution of a system of random operator equations. Let us first mention some definitions. Let $(\Omega, \mathcal{A}, \mu)$ be a complete probability measure space and let (X, d) be a metric space. By $\mathcal{B}$ we mean σ-algebra of Borel subsets of X, so that $(X, \mathcal{B})$ is a measurable space. A mapping $x : \Omega \to X$ is called an X-valued random variable (or generalized random variable), if $x^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$. A mapping $A : \Omega \times X \to X$ is said to be a random operator if, for any $x \in X$, $A(., x)$ is a random variable. A random operator A is continuous if, for each $\omega \in \Omega$, $A(\omega, .)$ is continuous in the topology induced by the metric d. The ordered pair $(E, \mathcal{F})$ is an E-space over (X, d) if the elements of E are equivalence classes of measurable functions from $(\Omega, \mathcal{A}, \mu)$ into X such that, for every $x, y \in E$ and $t \in \mathbb{R}$, the set $\{\omega \in \Omega : d(x(\omega), y(\omega)) < t\}$ belongs to $\mathcal{A}$, and $\mathcal{F}$ is given via $F_{x,y}(t) := \mu\{\omega \in \Omega : d(x(\omega), y(\omega)) < t\}$. By [7] it is known that $(E, \mathcal{F}, T_m)$ is a

Menger space. Moreover, if (X, d) is a complete metric space, then $(E, \mathcal{F}, T_m)$ is complete. A random variable $x(\omega) \in E$ is said to be a random fixed point of the random operator $A(\omega, .)$ if $x(\omega) = A(\omega, x(\omega))$, $\forall \omega \in \Omega$. If A is continuous, then $A(\omega, x(\omega)) \in E$, whenever $x(\omega) \in E$. Now, we assume $(X, |\cdot|)$ is a Banach space: $d(x,y) := |x-y|$. Consider the following system of random operator equations

$$\begin{cases} x(\omega) = A_1(\omega, x(\omega)) + \alpha_1(\omega) \\ y(\omega) = A_2(\omega, y(\omega)) + \alpha_2(\omega) \\ u(\omega) = B_1(\omega, u(\omega)) + \beta_1(\omega) \\ v(\omega) = B_2(\omega, v(\omega)) + \beta_2(\omega), \end{cases} \tag{3.4}$$

where $\alpha_i, \beta_i \in E$, $i = 1, 2$. We define $f_i, h_i : E \to E$ by putting $(f_i x)(\omega) := A_i(\omega, x(\omega)) + \alpha_i(\omega)$, $(h_i x)(\omega) := B_i(\omega, x(\omega)) + \beta_i(\omega)$, $i = 1, 2$.

Theorem 3.3. *Let $(\Omega, \mathcal{A}, \mu), (X, |\cdot|), (E, \mathcal{F}, T_m), A_i, B_i, \alpha_i, \beta_i, f_i, h_i$, $i = 1, 2$ be as above. Assume*
(a) $f_1(E) \subset h_2(E)$, $f_2(E) \subset h_1(E)$,
(b) f_1, h_1 are probabilistic compatible, and so are f_2, h_2,
(c) one of f_1, f_2, h_1, h_2 is continuous,
(d) there exists a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition (G) *such that, for all x, y in E and $t > 0$,*

$$\begin{aligned} &\mu\{\omega \in \Omega : |(f_1 x)(\omega) - (f_2 y)(\omega)| \geq g(t)\} \\ &\quad \leq g(\max\{\mu\{\omega \in \Omega : |(h_1 x)(\omega) - (h_2 y)(\omega)| \geq t\}, \\ &\qquad \mu\{\omega \in \Omega : |(h_1 x)(\omega) - (f_1 x)(\omega)| \geq t\}, \\ &\qquad \mu\{\omega \in \Omega : |(h_2 y)(\omega) - (f_2 y)(\omega)| \geq t\}, \\ &\qquad \frac{1}{2}[\mu\{\omega \in \Omega : |(h_1 x)(\omega) - (f_2 y)(\omega)| \geq t\} \\ &\qquad + \mu\{\omega \in \Omega : |(h_2 y)(\omega) - (f_1 x)(\omega)| \geq t\}]\}). \end{aligned} \tag{3.5}$$

Then there exists a unique solution of the system (3.4).

Proof. This follows from Corollary 3.2, since (3.5) is equivalent to (3.3).

Remark 4. As noted after Corollary 3.2, the same conclusion of Theorem 3.5 (but under stronger conditions) was obtained in Theorem 4.1 of [2], as an immediate consequence of Theorem 3.2 of [2].

Corollary 3.4. *In the notation above, if $h_1(\omega, x) = h_2(\omega, x) \equiv x$, and* (3.5) *holds for all $x \neq y$ in E, then at least one of the first two equations of* (3.4) *has a solution. If both of them have solutions, then they have a unique common solution which is also a unique solution for each of them.*

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References

1. T.-H. Chang, Common fixed point theorems in Menger spaces, *Bull. Inst. Math. Acad. Sinica* **22** (1994) 17–29.
2. Y. J. Cho, K. S. Ha, and S.-S. Chang, Common fixed point theorems for compatible mappings of type (A) in non-Archimedean Menger PM-spaces, *Math. Japon.* **46** (1997) 169–179.
3. Do Hong Tan, Some common fixed point theorems for mappings of contractive type, *Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak., Ser. Mat.* **25** (2) (1995) 9–22.
4. A. Meir and E. Keeler, A theorem on contractive mappings, *J. Math. Anal. Appl.* **28** (1969) 326–329.
5. K. Menger, Statistical metrics, *Proc. Nat. Acad. Sci. USA* **28** (1942) 535–537.
6. B. Schweizer and A. Sklar, Statistical metric spaces, *Pacific J. Math.* **10** (1960) 313–334.
7. H. Sherwood, On E-spaces and their relation to other classes of probabilistic metric spaces, *J. London Math. Soc.* **44** (1969) 441–448.
8. R. R. Stevens, Metrically generated probabilistic metric spaces, *Fund. Math.* **LXI** (1968) 259–269.
9. C. S. Wong, Common fixed points of two mappings, *Pacific J. Math.* **48** (1973) 299–312.
10. C. S. Wong, Generalized contractions and fixed point theorems, *Proc. Amer. Math. Soc.* **42** (1974) 409–417.