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Short Communication

On the Construction of Initial Polyhedral Convex Set for Optimization Problems Over the Efficient Set and Bilevel Linear Programs*

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1. Introduction

Two basic techniques widely used in global optimization are branch-and-bound and outer approximations. Algorithms using these techniques require constructing at the beginning an initial polyhedral convex set whose vertices and extreme directions are easy to calculate. Moreover this set must contain at least one optimal solution and not beyond the domain where the objective function as well as constraints are defined. Throughout this note we mean such a polyhedral convex set as a *well initiated polyhedron*. When the objective function and the constraints of the problem are finite on the whole space, a well-initiated polyhedron can be constructed by available methods. Usually it is a simplex, a cone,or a box depending on the structure of the considered problem. There are however some important problems for which finding a well-nitiated polyhedron is not an easy task, because the objective and/or constraint functions are not defined everywhere or their effective domains are not given explicitly. Examples for such problems can be taken from a class of multiplicative optimization, bilevel programming, and optimization over the efficient set.

In this note we propose the use of outer and inner (primal and dual) approximations, which are widely used in global optimization, to construct a well initiated polyhedron for optimization problems over the efficient set of a multiple objective linear program and for bilevel linear programs. Branch-and-bound

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methods using an initial polyhedron constructed according to our proposal require special subdivision strategies because the resulting initial polyhedron does not, in general, belong to the sets of cones, simplices, and boxes. For this case we make use of an adaptive polyhedral subdivision developed in convex-concave programming to obtain a decomposition branch-and-bound algorithm for solving bilevel linear progams.

2. Examples

Below are examples for which constructing a well-initiated polyhedron, which is a cone, a simplex, or a box, is not an easy task.

(1) *Positive Multiplicative Optimization* [3]

Consider the following multiplicative programming problem:

$$\min\{f(x) := \Pi_{i=1}^{k} f_i(x) : x \in D\}. \tag{1}$$

Suppose that D is a closed convex set and f_i ($i = 1, ..., k$) is affine, positive valued on D. Then $\Pi_{i=1}^{k} f_i(x)$ is quasiconcave on D. So in this case, Problem (1) is a quasiconcave minimization. Note that without the restriction to positive values of f_i, $i = 1, ..., k$, the quasiconcavity property of f may fail to hold. Therefore, solution methods rely on quasiconcavity of the objective function should work within a domain where all f_i $(i = 1, ..., k)$ are positive valued.

(2) *Optimization Over the Efficient Set* [1, 2, 4, 6, 7]

Let X be a bounded polyhedron in R^n, C a $(p \times n)$-matrix, and f a real-valued function on R^n. Let $E(C, X)$ denote the set of all efficient (Pareto) points of C on X, i.e.,

$$E(C, X) := \{x \in X : Cy \geq Cx, y \in X \Rightarrow Cx = Cy\}.$$

The problem of finding a most preferred (with respect to f) efficient point can be written as

$$\max\{f(x) : x \in E(C, X)\}. \tag{2}$$

Let

$$C_0 := \{x \in R^n : Cx \leq 0\},\ G(X) := \{x \in R^n : Cy \geq Cx,\ y \in X\}$$

and

$$r(x) := \max\{e^T(Cy - Cx) : Cy \geq Cx, y \in X\},$$

where as usual e denotes the vector whose every entry is 1. It has been shown [1, 4] that (2) is equivalent to the problem

$$\max\{f(x) : r(x) \leq 0, x \in X\}. \tag{3}$$

Since the effective domain $G(X)$ of r is not given explicitly, constructing a well initiated polyhedron contained in $G(X)$ is not straightforward.

(3) *Bilevel Linear Programming* [9,10]

Let K and L be two bounded polyhedral convex sets in R^p and R^q respectively, and let $f : R^p \times R^q \to R$, $g : R^q \to R$ be linear functions. Consider the following bilevel linear program:

$$\max f(x,y) \text{ s.t. } x \in K \text{ and } y \text{ solves} \tag{4}$$

$$\max\{g(y) : y \in L, Ax + By \leq r\}, \tag{$P(x)$}$$

where $r \in R^m$, and A, B are given appropriate matrices. Let

$$p_0(x) := \max\{g(y) : y \in L, By \leq r - Ax\}. \tag{5}$$

Since g is linear, p_0 is a finite concave function on the set

$$G(L) := \{x : By \leq r - Ax, y \in L\}.$$

As usual we assume that, for every $x \in K$, Problem $(P(x))$ has a feasible solution. Thus, $K \subset G(L)$. For each $z = (x,y) \in G(L) \times R^q$, define $p(x,y) := p_0(x) - g(y)$. Then p is a finite concave function on $G(L) \times R^q$. Clearly,

$$(x,y) \text{ is feasible for (4) if and only if } (x,y) \in K \times L, Ax + By \leq r, p(x,y) = 0. \tag{6}$$

Thus, Problem (4) can be formulated as

$$\max\{f(x,y) : x \in K, y \in L, Ax + By \leq r, p(x,y) \leq 0\}. \tag{7}$$

3. Construction of an Initial Polyhedral Convex Set to Optimization Over the Efficient Set and Bilevel Linear Programming

In branch-and-bound and outer approximation methods we are often given two polyhedra P and Q satisfying $P \subset Q$. It requires us to construct a polyhedron S such that $P \subset S \subset Q$ and its extreme points and directions can be calculated with a reasonable effort. Below we propose two procedures using inner and outer approximations for constructing such a polyhedron. In the sequel we assume that P is a bounded polyhedron (polytope) given by a finite system of inequalities.

3.1. Outer Approximation

Suppose that a bounded polyhedral convex set S_0 containing P has been constructed. If $S_0 \subset Q$, we are done. Otherwise, there must exist a vertex v of S_0 such that $v \notin Q$. Then $v \notin P$, since $P \subset Q$. Find a constraint of P violated by v. Add this constraint to S_0 to obtain a new polytope S_1. Then the procedure is repeated with S_1 and so on. Since $P \subset Q$, the procedure must terminate yielding a polytope containing P and contained in Q. Since searching vertices is very costly in high-dimensional spaces, we suggest reducing, if possible, the dimension of the space in which this searching takes place. Let us illustrate this procedure by applying it to the feasible domain of Problem (3) which is given as $D := \{x : r(x) \leq 0, x \in X\}$. For simplicity, assume that rank $C = k$, that the first k rows $c^1, ..., c^k$ of C are independent, and that all the data are given in the basis $c^1, ..., c^k, b^{k+1}, ..., b^n$, where $b^{k+1}, ..., b^n$ forms a basis of the subspace $L_2 := \{x : Cx = 0\}$. Denote by L_1 the subspace generated by $\{c^1, ..., c^k\}$. Then every x is uniquely expressed as $x = x^1 + x^2$ with $x^1 \in L_1$, $x^2 \in L_2$. Thus, every x is uniquely defined by a couple of vectors (u, v), where $u = (u_1, ..., u_k)$ and $v = (v_{k+1}, ..., v_n)$ are calculated by the following system of equations:

$$x^1 = \sum_{j=1}^{k} u_j c^j, \quad x^2 = \sum_{j=k+1}^{n} v_j b^j.$$

It has been shown (see e.g., [4]) that $r(x) = r(x^1) = r(u)$. Thus, the constraint $r(x) \leq 0$ that makes D non-convex actually only depends upon k-variables. This allows us to apply the above outer approximation in L_1. Namely, we take $P = X_1$, and $Q = G(X_1)$ where X_1 stands for the projection of X on L_1 and $G(X_1)$ is the effective domain of $r(u)$.

3.2. Inner Approximation

In inner approximation, which can be regarded as a dual form of the above outer approximation, we suppose that a polyhedron S_0 containing the origin and contained in Q has been defined and that the vertices and extreme directions of its polar, denoted by S_0^*, can be easily computed. If $P \subset S_0$, then we are done. Otherwise, there exists $v \in S_0^*$ such that

$$\langle v, a \rangle > 1 \text{ with } a \in \operatorname{argmax}\{\langle v, x \rangle : x \in V(P)\} \qquad (L(v))$$

($V(P)$ stands for the set of the vertices of P). Then we set $S_1 = \operatorname{conv}(S_0, \{a\})$ and repeat the procedure with S_1. Clearly $S_1 \subset Q$. Since the number of vertices of P is finite, this procedure must terminate yielding a polyhedron containing P and contained in Q.

Since $S_1 = \operatorname{conv}(S_0, \{a\})$, we have $S_1^* = S_0^* \cap \{u : \langle a, u \rangle \leq 1\}$. Since $\dim S_0^*$ is equal to n-linearityS_0 and the latter is a measure of non-linearity of S_0, this inner (dual outer) approximation procedure is expected to apply to problems where S_0 has low non-linearity.

To illustrate the proposed methods, let us apply them to the optimization problem over the efficient set and to the bilevel program given in the previous section.

Consider first the optimization problem over the efficient set in the form of (3). Without loss of generality, we assume that X contains the origin. Then $C_0 \subset G(X)$. Since $C_0 = \{x : Cx \leq 0\}$, its polar is

$$C_0^* = \{u : u = \sum_{j=1}^{k} t_j c^j, t_j \geq 0 \; \forall \; j = 1, ..., k\}.$$

We can then apply the above proposed inner approximation method with $P = X$, $Q = G(X)$, and $S_0 = C_0$. Since $\dim S_0^* = k$, the vertices and extreme directions are created in a k-dimensional space.

Next we consider the bilevel program given by (7). As before, assume that the origin is feasible. We observe that $p_0(x)$ is constant on the linear space $L_0 := \{x : Ax = 0\}$. Hence, the function $p(x,y) = p_0(x) - g(y)$ is convex on L_0. Since the origin is feasible for (7), we have $C_0 := \{x : Ax \leq 0\} \subset G(L)$. We then can apply the inner approximation method with $P = K$, $Q = G(L)$ and $S_0 = C_0$. Note that if $\text{rank}A = k$, then the vertices and extreme directions, are computed, as before, in a k-dimensional space.

4. On Solution Method by Branch-and-Bound

The initial polyhedral convex set constructed by the above methods in general is neither a simplex, a cone, nor a rectangle. Therefore, branch-and-bound methods using simplicial, conical, or rectangular subdivisions, in general, cannot be used for this set. We propose to use an adaptive polyhedral bisection developed in convex-concave programming to obtain a decomposition algorithm for solving Problem (7).

Suppose that a polyhedral convex set S_0 satisfying $K \subset S_0 \subset G(L)$ has been constructed. Let S be a subpolyhedron of S_0 whose vertices $V(S)$ and extreme directions $R(S)$ are at hand. Consider the bilevel program (7) with respect to S, i.e.,

$$\beta(S) := \max\{f(x,y) : x \in S, y \in L, Ax + By \leq r, p_0(x) - g(y) \leq 0\} \qquad (P(S))$$

By decoupling variables x and y in $(P(S))$, we obtain the relaxed problem

$$\alpha(S) := \max\{f(x,y) : x \in S \cap K, u \in S, y \in L, \\ Ax + By \leq r, p_0(u) - g(y) \leq 0\}. \qquad (R(S))$$

Since p_0 is finite and concave on S, it is easy to see that

$$\alpha(S) = \max\{f(x,y) : x \in S \cap K, y \in L, \\ Ax + By \leq r, p_0(u^S) - g(y) \leq 0\}, \qquad (P(u^S))$$

where

$$u^S \in \arg\min\{p_0(u) : u \in V(S)\}.$$

Note that since f and g are linear, $(P(u^S))$ is a linear program.

Let (x^S, y^S) be an optimal solution of Problem $(P(u^S))$. Clearly, $\beta(S) \leq \alpha(S)$, and $\beta(S) = \alpha(S)$ whenever $u^S = x^S$. Thus, if $u^S \neq x^S$, one suggests bisecting S into two polyhedral convex sets by a separation function defined by a hyperplane passing the midpoint of segment joining u^S and x^S. Namely, we set

$$S_1 := \{x \in S : \langle l^S, x - (u^S + x^S)/2 \rangle \leq 0\},$$
$$S_2 := \{x \in S : \langle l^S, x - (u^S + x^S)/2 \rangle \geq 0\},$$

where $l^S = (u^S - x^S)/(\|u^S - x^S\|)$. The points x^S and u^S are called *bisection points* of S. As usual, a partition set S_k is selected to be bisected at iteration k if

$$\alpha(S_k) = \max\{\alpha(S) : S \in \mathcal{R}_k\},$$

where $\mathcal{R}_k$ denotes the family of partition sets of interest at iteration k.

It has been shown [5] that if $\{S_j\}$ is an infinite nested sequence of partition sets generated by this bisection and $\{u^j\}$, $\{x^j\}$ are two sequences of corresponding bisection points, then the sequences $\{u^j\}$, $\{x^j\}$ have a common cluster point. This property ensures that a branch-and-bound algorithm using the just described bounding, branching, and selecting rules will be convergent in the sense that the algorithm produces a non-increasing sequence of upper bounds and a non-decreasing sequence of lower bounds which both tend to the global optimal value of the problem. Moreover, the common cluster point of bisection points of selected partition sets is a global optimal solution.

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