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Remarks on the Schauder–Tychonoff Fixed Point Theorem

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Abstract. We give a simple proof of a generalization of a Schauder–Tychonoff type fixed point theorem directly using the KKM principle.

1. Introduction

The celebrated KKM principle due to Knaster, Kuratowski, and Mazurkiewicz [8] was used to prove the Brouwer fixed point theorem (see also [11, 12]). The converse is now well known (see [1]). The Brouwer theorem is generalized to normed vector spaces by Schauder, and to locally convex topological vector spaces by Tychonoff. These are all unified by Hukuhara [6]. Moreover, Ky Fan [3] deduced the Tychonoff fixed point theorem from his own generalization of the KKM principle, so called the KKMF principle.

In this note we give a simple proof of a generalization of the Schauder–Tychonoff-type fixed point theorem for compact maps in locally convex topological vector spaces directly using the KKM principle. Finally, several remarks on our results and our proof are added.

2. Preliminaries

Before establishing the result we recall the KKM principle [8]:

KKM principle. *Let D be the set of vertices of a simplex S and $F : D \to 2^S$ a multimap with closed values such that*

$$\operatorname{co} N \subset F(N) \quad \text{for each } N \subset D.$$

Then $\bigcap_{z\in D} F(z) \neq \emptyset$.

Here, "co" stands for the convex hull of a set, and $F(N) = \bigcup_{z \in N} F(z)$.
From the principle we obtain immediately the following due to Fan [3]:

Lemma. *Let X be a subset of a topological vector space, D a non-empty finite subset of X such that $\operatorname{co} D \subset X$, and $F : D \to 2^X$ a multimap with closed values such that*

$$\operatorname{co} N \subset F(N) \quad \text{for each } N \subset D. \tag{1}$$

Then $\bigcap_{z \in D} F(z) \neq \emptyset$.

Recall that a map satisfying condition (1) is called a *KKM map*. We need also the following notion due to Himmelberg [5]:

Definition. *A non-empty subset X of a topological vector space E is said to be almost convex if, for any neighborhood V of the origin 0 in E and for any finite set $\{x_1, \dots, x_n\} \subset X$, there exists a finite set $\{z_1, \dots, z_n\} \subset X$ such that, for each $i \in \{1, \dots, n\}$, $z_i - x_i \in V$ and $\operatorname{co}\{z_1, \dots, z_n\} \subset X$.*

Clearly, each convex set is almost convex, but the converse is not true in general.

3. Results

Now we are in a position to prove the following result directly using the KKM principle.

Theorem 1. *Let X be an almost convex subset of a locally convex Hausdorff topological vector space E and $f : X \to X$ a compact continuous map. Then f has a fixed point.*

Proof. For any neighborhood U of 0 in E, there exists a symmetric open neighborhood V of 0 such that $V + V \subset U$. Since f is a compact map, $K := \overline{f(X)}$ is a compact subset of X. Hence, there exists a subset $\{x_1, x_2, \dots, x_n\}$ of K such that $K \subset \bigcup_{i=1}^n (x_i + V)$. Since X is almost convex, there exists a subset $D = \{z_1, z_2, \dots, z_n\}$ of X such that $z_i - x_i \in V$ for each $i = 1, 2, \dots, n$, and $\operatorname{co}\{z_1, z_2, \dots, z_n\} \subset X$. Let L be the finite-dimensional subspace of E generated by D.

For each i, let

$$F(z_i) := \{x \in X : f(x) \notin x_i + V\}.$$

Since V is open and f is continuous, each $F(z_i)$ is closed in X. Then

$$\bigcap_{i=1}^n F(z_i) = \{x \in X : f(x) \notin \bigcup_{i=1}^n (x_i + V)\} = \emptyset$$

since $f(X) \subset K \subset \bigcup_{i=1}^n (x_i + V)$. Now, we apply the Lemma to X with D defined as above. Since its conclusion does not hold, $F : D \to 2^X$ cannot be a KKM map. That is, there exist a subset $\{z_{i_1}, z_{i_2}, \ldots, z_{i_k}\}$ of $\{z_1, z_2, \ldots, z_n\}$ and an $x_U \in \mathrm{co}\{z_{i_1}, z_{i_2}, \ldots, z_{i_k}\}$ such that $x_U \notin \bigcup_{j=1}^{k} F(z_{i_j})$. Hence, for all $j = 1, 2, \ldots, k, f(x_U) \in x_{i_j} + V$, and

$$f(x_U) \in x_{i_j} + V = x_{i_j} - z_{i_j} + z_{i_j} + V \subset z_{i_j} + V + V \subset z_{i_j} + U. \qquad (2)$$

Denote

$$M := \{y \in L : f(x_U) \in y + U\}.$$

Since E is locally convex, we may assume that U is convex. Besides, since L is a vector subspace of E, it is easy to verify that M is convex too. From (2) we obtain that $z_{i_j} \in M$ for $j = 1, 2, \ldots, k$ and hence

$$\mathrm{co}\{z_{i_1}, z_{i_2}, \ldots, z_{i_k}\} \subset M$$

because M is convex. Therefore, $x_U \in M$.

So, for each neighborhood U of 0, there exist $x_U, y_U \in X$ such that $y_U = f(x_U)$ and $y_U \in x_U + U$. Since $f(X)$ is relatively compact, we may assume that the net $\{y_U\}$ converges to some $x_0 \in K$. Since E is Hausdorff, the net $\{x_U\}$ also converges to x_0. Since f is continuous, we have $x_0 = f(x_0)$. This completes our proof.

From Theorem 1, we have the following:

Theorem 2. *Let X be a non-empty convex subset of a locally convex Hausdorff topological vector space E and $f : X \to X$ a compact continuous map. Then f has a fixed point.*

4. Remarks

(1) It is well known that the Brouwer theorem, the Sperner lemma, and the KKM principle are equivalent (see [9]). For elementary proofs of the Sperner lemma and the KKM principle, see [12]. Combining these proofs with the proof of Theorem 1, we get an easier proof of the Brouwer theorem. For other proofs, see [9] and references therein.

(2) If X is a compact subset of an Euclidean space, Theorem 2 reduces to the Brouwer theorem. Theorem 2 is first due to Hukuhara [6] and is usually called the Schauder–Tychonoff fixed point theorem (see [1, 2]). In fact, if E is a normed vector space, Theorem 2 improves the original version of the Schauder theorem. Moreover, if X itself is compact, Theorem 2 reduces to the Tychonoff theorem (see [1] or [9]).

(3) There are a lot of generalizations of Theorem 2; for instance [4, 7, 10]. Note that Theorem 1 is a simple consequence of Theorem 4.3 of [7]. However, our aim is to give a proof directly from the KKM principle. The readers are kindly asked to compare our proof with that of the Schauder theorem in various text books (see [12]).

(4) Since Theorem 2 readily implies the Brouwer theorem, we conclude that the KKM principle, the Lemma (the KKMF principle), and the fixed point theorems due to Schauder and Tychonoff are all equivalent to Theorem 1.

(5) It is natural to ask whether the proof of Theorem 1 works for a Hausdorff topological vector space E not necessarily locally convex. In fact, even if X is convex, the proof heavily depends on the convexity of the set

$$M = \{y \in X \cap L : f(x_U) \in y + U\} = (f(x_U) + U) \cap X \cap L,$$

where U is a symmetric neighborhood of 0 in E, and L the finite-dimensional subspace of E generated by D.

Therefore, the convexity of M is always assured whenever E satisfies the following condition:

(∗) every neighborhood of 0 in E contains a neighborhood U of 0 such that $(x + U) \cap L$ is convex for all x in E and for any finite-dimensional subspace L of E.

If E is locally convex, then condition (∗) holds. The converse also holds. Indeed, take any $x_1, x_2 \in U$, $\lambda \in [0, 1]$ and denote $x_\lambda = \lambda x_1 + (1 - \lambda)x_2$. Further, take an arbitrary $x \in E$ and let L be the subspace of E generated by x, x_1, x_2. Since $x_i \in U \cap L$, $i = 1, 2$, and $x \in L$, we obtain

$$x + x_i \in (x + U) \cap L, \; i = 1, 2.$$

By hypothesis, $(x + U) \cap L$ is convex, so we have

$$x + x_\lambda = \lambda(x + x_1) + (1 - \lambda)(x + x_2) \in (x + U) \cap L.$$

From this we get $x + x_\lambda \in x + U$ and hence $x_\lambda \in U$. So U is convex, as claimed above.

This shows that, in our proof of Theorem 1, the local convexity of E (or, at least, certain local convexity of X; see Remark 6 below) is essential.

(6) Analyzing the proof of Theorem 1, we find that the crucial points in the proof are: $z_{i_j} \in M$ and M is convex. Since $z_{i_j} \in X$, we may define $M = \{y \in X : f(x_U) \in y + U\}$, where U can be supposed symmetric, so M can be rewritten as $M = \{y \in X : y \in f(x_U) + U\} = (f(x_U) + U) \cap X$. Since $f(x_U) \in X$, for convexity of M, it suffices to require

$$(x + U) \cap X \text{ is convex for all } x \in X. \tag{3}$$

So the proof of Theorem 1 works if the following condition holds:

(A) Every neighborhood V of $0 \in E$ contains a neighborhood U of $0 \in E$ such that (3) holds.

Therefore, in Theorem 1, *we can replace the local convexity of* E *by the condition* (A) *on* X. Now we look for the meaning of condition (A). Since every neighborhood of $x \in X$ in the induced topology has the form $(x + V) \cap X$ with a neighborhood V of $0 \in E$, the convex neighborhood $(x + U) \cap X \subset (x + V) \cap X$

shows that the induced topology in X is locally convex; in other words, X becomes a locally convex subset of E (in the sense of Krauthausen; see [4, p. 26]). Note that Krauthausen gave a number of examples of locally convex subsets of non-locally convex spaces.

(7) We may consider another type of condition on X as follows:

(B) For each $x \in X$ and each neighborhood V of $0 \in E$, there exists a neighborhood U of $0 \in E$ such that

$$\mathrm{co}((x+U)\cap X) \subset x+V. \tag{4}$$

Note that (A) implies (B) because

$$\mathrm{co}((x+U)\cap X) = (x+U)\cap X \subset (x+V)\cap X \subset x+V.$$

The converse holds if X is convex. Indeed, since $(x+U)\cap X \subset X$ and X is convex, we get $\mathrm{co}((x+U)\cap X) \subset X$, which together with (4) gives $\mathrm{co}((x+U)\cap X) \subset (x+V)\cap X$. Since $(x+U)\cap X$ is a neighborhood of $x \in X$ and $(x+U)\cap X \subset \mathrm{co}((x+U)\cap X) \subset X$, $\mathrm{co}((x+U)\cap X)$ is a convex neighborhood of $x \in X$ which is contained in $(x+V)\cap X$.

Therefore, our proof also works for Theorem 2 under condition (B) instead of local convexity of E.

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References

1. J. Dugundji and A. Granas, *Fixed Point Theory*, Polish Scientific Publishers, Warszawa, 1982.
2. N. Dunford and J. T. Schwartz, *Linear Operators, Part I: General Theory*, Interscience Publishers, New York, 1957.
3. K. Fan, A generalization of Tychonoff's fixed point theorem, *Math. Ann.* **142** (1961) 305–310.
4. O. Hadžić, *Fixed Point Theory in Topological Vector Spaces*, University of Novi Sad, Novi Sad, 1984, 377 pp.
5. C. J. Himmelberg, Fixed points of compact multifunctions, *J. Math. Anal. Appl.* **38** (1972) 205–207.
6. M. Hukuhara, Sur l'existence des points invariants d'une transformation dans l'espace fonctionnel, *Jap. J. Math.* **20** (1950) 1–4.
7. A. Idzik, Almost fixed point theorems, *Proc. Amer. Math. Soc.* **104** (1988) 779–784.
8. B. Knaster, K. Kuratowski, and S. Mazurkiewicz, Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe, *Fund. Math.* **14** (1929) 132–137.

9. S. Park, Eighty years of the Brouwer fixed point theorem, in: *Antipodal Points and Fixed Points*, J. Jaworowski, W. A. Kirk, and S. Park (eds.), Lecture Notes Series, Vol. 28, RIM-GARC, Seoul National University, 1995, pp. 55–97.
10. S. Park, A unified fixed point theory of multimaps on topological vector spaces, *J. Korean Math. Soc.* **35** (1998) 803–829.
11. E. Zeidler, *Nonlinear Functional Analysis and Its Applications, I. Fixed Point Theorems*, Teubner-Verlag, Leibzig, 1976.
12. E. Zeidler, *Applied Functional Analysis – Applications to Mathematical Physics*, Springer-Verlag, New York, 1995.