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On Weak Convergence of the Bootstrap Empirical Process with Random Resample Size*

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Abstract. In this paper we obtain the weak convergence of the bootstrap empirical process with random resample size. The proof is based on the use of dual Lipschitz metric, defined by the weak topology on the space of probability measures on D, where D is the space of all real-valued functions f on $[-\infty,\infty]$, such that f vanishes continuously at $\pm\infty$ and is right continuous with left limits on $(-\infty,\infty)$.

1. Introduction

Consider a sequence $\{X_i, i \geq 1\}$ of independent and identically distributed stochastic variables, defined on a probability space $(\Omega, \mathcal{A}, P)$, with each X_i having a distribution function F. Let $F_n(t)$ be the empirical distribution function of $(X_1, \ldots, X_n)$, and set

$$W_n(t) = \sqrt{n}\{F_n(t) - F(t)\} \quad \text{for } -\infty < t < \infty,$$

extended to vanish at $\pm\infty$.

Let D be the space of all real-valued functions f on $[-\infty, \infty]$, such that f vanishes continuously at $\pm\infty$, and is right continuous with left limits on $(-\infty, \infty)$. Give D the Skorokhod topology. Let $\psi_n(F)$ be the distribution of the process W_n. Thus, $\psi_n(F)$ is a probability measure on D. In this notation, the usual invariance principle states that $\psi_n(F)$ tends weakly to the law of $B(F)$ as $n \to \infty$, where B is the Brownian bridge on $[0, 1]$, and $B(F)(t, \omega) = B\{F(t), \omega\}$.

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Now, let $\{N_n, n \geq 1\}$ be a sequence of positive integer-valued random variables, such that

$$\frac{N_n}{n} \to_p \nu \quad \text{as } n \to \infty, \tag{1.1}$$

where ν is a positive random variable defined on the same probability space $(\Omega, \mathcal{A}, P)$ and $\to_p$ denotes convergence in probability. Pyke [5], Fernander [3] (for $\nu = 1$), Csörgö [2], Sen [6] (for $\nu > 0$ arbitrary) have shown that under (1.1), W_{N_n} converges in law to $B(F)$. The aim of the present paper is to bootstrap the empirical process with random sample size. Suppose that there is a sample of size n from an unknown F, which is to be estimated by the empirical distribution function F_n. Given $X_1, \dots, X_n$, let $X^*_{n1}, \dots, X^*_{nm}$ be conditionally independent, with common distribution F_n. Let $F^*_{m,n}$ be the empirical distribution function of $X^*_{n1}, \dots, X^*_{nm}$, and let

$$W^*_{m,n}(t) = \sqrt{m}\{F^*_{m,n}(t) - F_n(t)\} \quad \text{for } -\infty < t < \infty,$$

extended to vanish at $\pm\infty$. Thus, $X^*_{n1}, \dots, X^*_{nm}$ is the "bootstrap sample", $F^*_{m,n}$ is the "bootstrap empirical distribution function", and $W^*_{m,n}$ is the "bootstrap empirical process".

Bickel and Freedman [1] have allowed the resample size m to differ from the number n of data points and proved that, for almost all sample sequences $X_1, \dots, X_n, \dots,$ $W^*_{m,n}$ converges weakly to $B(F)$ as $m \wedge n \to \infty$.

Here we are first concerned with the weak convergence of the process $W^*_{N_n} = W^*_{N_n,n}$, where $\{N_n, n \geq 1\}$ is a sequence of positive integer-valued random variables such that (1.1) holds. In the case where N_n is independent of the sequence $X_1, X_2, \dots$, we show that the bootstrap works for the empirical process if the random bootstrap sample size N_n is such that

$$N_n \to_p \infty \quad \text{as } n \to \infty. \tag{1.2}$$

2. Results

Theorem 2.1. *If the sequence of positive integer-valued random variables N_n is such that (1.1) holds, then along almost all sample sequences $X_1, X_2, \dots$, given $(X_1, \dots, X_n)$, $W^*_{N_n}$ converges weakly to $B(F)$ as $n \to \infty$. Here, $W^*_{N_n} = W^*_{N_n,n}$ and $W^*_{m,n}$ is the bootstrap empirical process, as defined above.*

Corollary 2.1. *If the sequence of positive integer-valued random variables N_n is such that (1.1) holds, then along almost all sample sequences $X_1, X_2, \dots$, given $(X_1, \dots, X_n)$,*

$$\|F^*_{N_n} - F\| \to_p 0 \quad \text{as } n \to \infty.$$

Here, $\|F^*_{N_n} - F\| = \sup_{-\infty<t<\infty} |F^*_{N_n}(t) - F(t)|$, $F^*_{N_n} = F^*_{N_n,n}$ and $F^*_{m,n}$ is the empirical distribution of the resampled data as defined above.

In the case where N_n is independent of the sequence $X_1, X_2, \dots$, we have the following theorem.

Theorem 2.2. *If the sequence of positive integer-valued random variables N_n is independent of the sequence $X_1, X_2, \ldots$, and (1.2) holds, then along almost all sample sequences $X_1, X_2, \ldots$, given $(X_1, \ldots, X_n)$, $W^*_{N_n}$ converges weakly to $B(F)$ as $n \to \infty$.*

Corollary 2.2. *If the sequence of positive integer-valued random variables N_n is independent of the sequence $X_1, X_2, \ldots$, and (1.2) holds, then along almost all sample sequences $X_1, X_2, \ldots$, given $(X_1, \ldots, X_n)$,*

$$\|F^*_{N_n} - F\| \to_p 0 \quad \text{as } n \to \infty.$$

3. Proofs

We first recall the standard terminology and notation of weak convergence. Let $U_1, \ldots, U_n$ be a random sample taken from the uniform distribution on $[0,1]$ and let H_m be the empirical distribution function of $U_1, \ldots, U_n$ and

$$B_m(t) = \sqrt{m}\{H_m(t) - t\} \quad \text{for } 0 \le t \le 1.$$

Let D be the space of all real-valued functions f on $[-\infty, \infty]$, as defined in Sec. 1. The weak topology on the space of probability measures on D is metrized by a dual Lipschitz metric as follows. Let γ metrize the Skorokhod topology on D, and, in addition, satisfy

$$\gamma(f,g) \le \|f - g\| \wedge 1. \tag{3.1}$$

Here, f and g are elements of D, i.e., functions on $[-\infty, \infty]$, and $\|\,.\,\|$ is the sup norm. Now

$$\rho(\pi, \pi') = \sup_\theta \left| \int_D \theta d\pi - \int_D \theta d\pi' \right|, \tag{3.2}$$

where π and π' are probability measures on D, and θ runs through the functions on D which are uniformly bounded by 1 and satisfy the Lipschitz condition

$$\|\theta(f) - \theta(g)\| \le \gamma(f,g).$$

Proof of Theorem 2.1. Let $[s]$ denote the largest integer $\le s$. Let $\psi_{N_n}(F_n)$, $\psi_{[n\nu]}(F)$, and $\psi(F)$ be the distributions of the processes $W^*_{N_n}$, $W_{[n\nu]}$, and $B(F)$, respectively. Clearly, $\psi_{N_n}(F_n)$ and $\psi_{[n\nu]}(F)$ are the probability distributions induced on D by $B_{N_n}(F_n)$ and $B_{[n\nu]}(F)$, respectively. In this way we can bound the dual Lipschitz metric ρ by

$$\rho\left[\psi_{N_n}(F_n), \psi(F)\right] \le \rho\left[\psi_{N_n}(F_n), \psi_{[n\nu]}(F)\right] + \rho\left[\psi_{[n\nu]}(F), \psi(F)\right]. \tag{3.3}$$

By (3.7) of [6], we have

$$\rho\left[\psi_{[n\nu]}(F), \psi(F)\right] \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

By the definition (3.2) of the dual Lipschitz metric ρ,

$$\begin{aligned}\rho\left[\psi_{N_n}(F_n), \psi_{[n\nu]}(F)\right] &\le \sup_{\theta} E\left\{\left|\theta\left[B_{N_n}(F_n)\right] - \theta\left[B_{[n\nu]}(F)\right]\right|\right\} \\ &\le E\left\{\gamma\left[B_{N_n}(F_n), B_{[n\nu]}(F)\right]\right\}.\end{aligned} \tag{3.5}$$

Now (3.1) implies

$$E\left\{\gamma\left[B_{N_n}(F_n), B_{[n\nu]}(F)\right]\right\} \le E\left\{\left\|B_{N_n}(F_n) - B_{[n\nu]}(F)\right\| \wedge 1\right\}.$$

Since $\|f - g\| \wedge 1$ is a metric, the triangle inequality implies

$$\begin{aligned}E\left\{\left\|B_{N_n}(F_n) - B_{[n\nu]}(F)\right\| \wedge 1\right\} &\le E\left\{\left\|B_{N_n} - B_{[n\nu]}\right\| \wedge 1\right\} \\ &\quad + E\left\{\omega\left(\|F_n - F\|, B_{[n\nu]}\right) \wedge 1\right\},\end{aligned} \tag{3.6}$$

where $\omega(\delta, f) = \sup\{|f(s) - f(t)| : |t - s| \le \delta\}$. By (3.6) of [6],

$$\left\|B_{N_n} - B_{[n\nu]}\right\| \wedge 1 \to_p 0.$$

Hence, by Corollary 3 of [4],

$$E\left\{\left\|B_{N_n} - B_{[n\nu]}\right\| \wedge 1\right\} \to 0. \tag{3.7}$$

By (3.3)–(3.7), it suffices to show that

$$E\left\{\omega\left(\|F_n - F\|, B_{[n\nu]}\right) \wedge 1\right\} \to 0 \quad \text{a.s.}$$

To show this, we note that, for every $\epsilon > 0$,

$$E\left\{\omega\left(\|F_n - F\|, B_{[n\nu]}\right) \wedge 1\right\} < \epsilon + P\left\{\omega\left(\|F_n - F\|, B_{[n\nu]}\right) \ge \epsilon\right\},$$

and hence it remains to prove that

$$P\left\{\omega\left(\|F_n - F\|, B_{[n\nu]}\right) \ge \epsilon\right\} \to 0 \quad \text{a.s.} \tag{3.8}$$

Since

$$\|F_n - F\| \to 0 \quad \text{a.s.} \tag{3.9}$$

by the Glivenko–Cantelli lemma, (3.8) follows from (3.8) of [4]. ■

Proof of Theorem 2.2. For the proof of Theorem 2.2 we need the following lemma.

Lemma 3.1. [1, Proposition 4.1] *Let F and G be probability distribution functions. Let $\psi_m(F)$ and $\psi_m(G)$ be the probability distributions induced on D by $B_m(F)$ and $B_m(G)$, respectively. Then there exists a universal constant C such that*

$$\rho\left[\psi_m(F), \psi_m(G)\right] \le C\left[\epsilon_m + h(\|F - G\|)\right],$$

where $\epsilon_m = m^{-1/2}\log m$ *and* h *is given by*

$$h(\delta) = \begin{cases} \sqrt{\delta \log \frac{1}{\delta}} & \textit{for } 0 \le \delta \le \dfrac{1}{2}, \\ h(\frac{1}{2}) & \textit{for } \delta \ge \dfrac{1}{2}. \end{cases}$$

To prove Theorem 2.2, it suffices by the triangle inequality for the dual Lipschitz metric ρ:

$$\rho\,[\psi_{N_n}(F_n), \psi(F)] \le \rho\,[\psi_{N_n}(F_n), \psi_{N_n}(F)] + \rho\,[\psi_{N_n}(F), \psi(F)]$$

and by Theorem 1 of [2]:

$$\rho\,[\psi_{N_n}(F), \psi(F)] \to 0$$

to show that

$$\rho\,[\psi_{N_n}(F_n), \psi_{N_n}(F)] \to 0 \quad \text{a.s.} \tag{3.10}$$

Now, use Lemma 3.1 and the fact that N_n and $X_1, X_2, \ldots$ are independent to estimate the term on the left in (3.10):

$$\begin{aligned} \rho\,[\psi_{N_n}(F_n), \psi_{N_n}(F)] &= \sum_{m=1}^{\infty} P[N_n = m]\rho\,[\psi_m(F_n), \psi_m(F)] \\ &\le \sum_{m=1}^{\infty} P[N_n = m] C[\epsilon_m + h(\|F_n - F\|)] \\ &= C[E(\epsilon_{N_n}) + h(\|F_n - F\|)]. \end{aligned}$$

From (3.9), we have

$$h(\|F_n - F\|) \to 0 \quad \text{a.s.}$$

Hence, (3.10) follows if we prove that

$$E(\epsilon_{N_n}) \to 0 \quad \text{as } n \to \infty. \tag{3.11}$$

For any $\eta > 0$ we can, by $\epsilon_m \to 0$, find M_η such that $\epsilon_m < \eta$ when $m > M_\eta$, and then

$$E(\epsilon_{N_n}) < \eta + P[N_n \le M_\eta]. \tag{3.12}$$

By (1.2) it follows that (3.12) can be made arbitrarily small by picking η small which proves (3.11). This completes the proof of Theorem 2.2. ∎

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