Vietnam Journal of Mathematics **28**:3 (2000) 257-262

Vietnam Journal of MATHEMATICS

An Iteration Scheme for Non-Expansive Mappings in Metric Spaces of Hyperbolic Type*

Nguyen Minh Chuong[1] **and Nguyen Van Co**[2]

[1] *Institute of Mathematics, P.O. Box 631, Bo Ho, Hanoi, Vietnam*

[2] *Department of Mathematics, Training Teachers College of Hanoi, Cau Giay, Hanoi, Vietnam*

Received September 23, 1999

Revised November 16, 1999

Abstract. An iteration scheme, more general and practically efficient than the one in [9], for non-expansive mappings in metric spaces of hyperbolic type is studied. Fixed point theorems are established.

1. Introduction

Iteration method is the most popular and powerful tool to solve approximately almost every kind of equations: differential, integral, partial differential, operator, linear and non-linear equations, especially the well-known iteration suggested by Krasnoselskii in [11]. After the publication of this work, many iteration processes of this kind are developed (see [1, 2, 4, 6] and references therein). In [9] Kirk has extended the Krasnoselskii iteration scheme to study fixed points of non-expansive mappings in metric spaces of hyperbolic type (see also [3, 8, 10, 12]).

The aim of this note is to develop this scheme. The developed scheme is not only more general than the one in [9], but also more practically efficient even in the holomorphic setting. Here the proof is also more complicated than the one in [9].

2. An Iteration Scheme in Metric Spaces of Hyperbolic Type

Let (X, d) be a metric space containing a family of metric lines such that distinct

* This paper was supported in part by the National Basic Research Program in Natural Sciences, Vietnam.

points $x, y \in X$ lie exactly on one member of the family. We denote by $M[x, y]$ the segment joining x and y.

The following condition H will be used:

Let $x, y, z \in X$, α be a number, $0 < \alpha < 1, m_1 \in M[x, y], m_2 \in M[x, z]$. If $d(x, m_1) = \alpha d(x, y)$ and $d(x, m_2) = \alpha d(x, z)$, then

$$d(m_1, m_2) \le \alpha d(y, z).$$

A metric space (X, d) satisfying the condition H is called a space of hyperbolic type.

Using condition H it is not difficult to prove the following:

Property 1. *Given $y, z \in X, 0 < \alpha < 1$. If $m \in M[y, z]$, and $d(y, m) = \alpha d(y, z)$, then*

$$d(x, m) \le \alpha d(x, z) + (1 - \alpha) d(x, y),$$

$\forall x \in X$.

Theorem 1. *Let (X, d) be a metric space of hyperbolic type, and $T : X \longrightarrow X$ to be non-expansive, i.e.,*

$$d(T(x), T(y)) \le d(x, y), x, y \in X. \tag{1}$$

Let x_0 be any point in X. The sequences $\{x_n\}, \{y_n\}, n \ge 0$, are defined as follows

$$\begin{cases} y_n = T(x_n), x_{n+1} \in M[x_n, y_n], \\ d(x_n, x_{n+1}) = \alpha_n d(x_n, y_n), 0 < \alpha_n < 1. \end{cases} \tag{2}$$

Then $\forall i, n \ge 0$ we have

$$\begin{aligned} d(x_i, y_{i+n}) \ge & \frac{1}{(1-\alpha_i)\cdots(1-\alpha_{i+n-1})}(a_{i+n} - a_i) \\ & + (1 + \alpha_i + \cdots + \alpha_{i+n-1}) a_i, \end{aligned} \tag{3}$$

where $a_k = d(x_k, y_k)$, (when $n = 0$ for (3) it is assumed that $d(x_i, y_i) = a_i$).

Before giving the proof of Theorem 1, some needed properties are stated without proofs which are quite easy. Let us denote by $\mathbb{N}$ the set of non-negative integers.

Property 2. *For the sequence defined by (2) we get*

$$d(y_{n+1}, y_n) \le d(x_{n+1}, x_n), \ \forall n \in \mathbb{N}.$$

Property 3. *For the sequence defined by (2) the inequality*

$$d(y_{n+1}, x_{n+1}) \le d(y_n, x_n), \forall n \in \mathbb{N}$$

holds true.

Proof of Theorem 1. We shall prove Theorem 1 by induction.

For any i, and $n = 1$, by Property 1,

$$d(x_{i+1}, y_{i+1}) \le \alpha_i d(y_i, y_{i+1}) + (1-\alpha_i) d(x_i, y_{i+1}).$$

By Property 2,

$$d(y_i, y_{i+1}) \le d(x_i, x_{i+1}) = \alpha_i d(x_i, y_i) = \alpha_i a_i.$$

Consequently,

$$a_{i+1} \le \alpha_i^2 a_i + (1-\alpha_i) d(x_i, y_{i+1})$$

and

$$(1-\alpha_i) d(x_i, y_{i+1}) \ge a_{i+1} - \alpha_i^2 a_i = (a_{i+1} - a_i) + (1-\alpha_i^2) a_i.$$

Because $1 - \alpha_i > 0$ we finally get

$$d(x_i, y_{i+1}) \ge \frac{1}{1-\alpha_i}(a_{i+1} - a_i) + (1+\alpha_i) a_i.$$

The inequality (3) is thus proved for any i and $n = 1$. Assume now that (3) is true for any i and for n. It will be proved that (3) is true for $n+1$ and for any i.

Indeed by Properties 1 and 2,

$$\begin{aligned} d(x_{i+1}, y_{i+1+n}) &\le \alpha_i d(y_i, y_{i+1+n}) + (1-\alpha_i) d(x_i, y_{i+1+n}) \\ &\le \alpha_i \big[d(y_i, y_{i+1}) + \cdots + d(y_{i+n}, y_{i+n+1})\big] + (1-\alpha_i) d(x_i, y_{i+1+n}) \\ &\le \alpha_i \big[d(x_i, x_{i+1}) + \cdots + d(x_{i+n}, x_{i+n+1})\big] + (1-\alpha_i) d(x_i, y_{i+1+n}) \\ &= \alpha_i \big[\alpha_i a_i + \alpha_{i+1} a_{i+1} + \cdots + \alpha_{i+n} a_{i+n}\big] + (1-\alpha_i) d(x_i, y_{i+1+n}). \end{aligned}$$

From this, using the induction assumption and Property 3, we get

$$\begin{aligned} d(x_i, y_{i+n+1}) &\ge \frac{1}{(1-\alpha_i)\cdots(1-\alpha_{i+n})}[a_{i+n+1} - a_{i+1}] \\ &\quad + \frac{1+\alpha_{i+1}+\cdots+\alpha_{i+n}}{1-\alpha_i} a_{i+1} - \frac{\alpha_i(\alpha_i + \cdots + \alpha_{i+n})}{1-\alpha_i} a_i \\ &= \frac{1}{(1-\alpha_i)\cdots(1-\alpha_{i+n})}[a_{i+n+1} - a_i] \\ &\quad + \left[\frac{1+\alpha_{i+1}+\cdots+\alpha_{i+n}}{1-\alpha_i} - \frac{1}{(1-\alpha_i)\cdots(1-\alpha_{i+n})}\right] a_{i+1} \\ &\quad + \left[\frac{1}{(1-\alpha_i)\cdots(1-\alpha_{i+n})} - \frac{\alpha_i(\alpha_i+\cdots+\alpha_{i+n})}{1-\alpha_i}\right] a_i. \end{aligned} \tag{4}$$

We stay here a moment to prove the following

Lemma 1. $\forall \alpha_{i+1}, \ldots, \alpha_{i+n} \in (0,1), \forall n \ge 1$ *the following inequality holds true:*

$$1 + \alpha_{i+1} + \cdots + \alpha_{i+n} - \frac{1}{(1-\alpha_{i+1})\cdots(1-\alpha_{i+n})} < 0. \tag{5}$$

Proof. Obviously, for $n = 1$, inequality (5) holds valid. Assuming that it is true for n, we shall prove that it is true for $n+1$.

Indeed, setting $\alpha_{i+n+1} = \alpha$, α is a variable in $(0,1)$, we consider the function

$$f(\alpha) = 1 + \alpha_{i+1} + \cdots + \alpha_{i+n} + \alpha - \frac{1}{(1-\alpha_{i+1})\cdots(1-\alpha_{i+n})(1-\alpha)}.$$

Noting that $f'(\alpha) < 0$ and by the induction assumption, we get

$$f(0) = 1 + \alpha_{i+1} + \cdots + \alpha_{i+n} - \frac{1}{(1-\alpha_{i+1})\cdots(1-\alpha_{i+n})} < 0.$$

So we have $f(\alpha) < f(0) < 0, (0 < \alpha < 1)$. Hence inequality (5) is valid for $n+1$.

We are now able to continue the proof of Theorem 1.

Taking into account that $a_{i+1} \leq a_i$, from (4) and Lemma 1, it follows that

$$\begin{aligned} d(x_i, y_{i+n+1}) \geq & \frac{1}{(1-\alpha_i)\cdots(1-\alpha_{i+n})}[a_{i+n+1} - a_i] \\ & + \left[\frac{1+\alpha_{i+1}+\cdots+\alpha_{i+n}}{1-\alpha_i} - \frac{1}{(1-\alpha_i)\cdots(1-\alpha_{i+n})}\right. \\ & \left. + \frac{1}{(1-\alpha_i)\cdots(1-\alpha_{i+n})} - \frac{\alpha_i(\alpha_i+\cdots+\alpha_{i+n})}{1-\alpha_i}\right]a_i \\ = & \frac{1}{(1-\alpha_i)\cdots(1-\alpha_{i+n})}[a_{i+n+1} - a_i] \\ & + \left[\frac{(1-\alpha_i^2)+\alpha_{i+1}(1-\alpha_i)+\cdots+\alpha_{i+n}(1-\alpha_i)}{1-\alpha_i}\right]a_i \\ = & \frac{1}{(1-\alpha_i)\cdots(1-\alpha_{i+n})}[a_{i+n+1} - a_i] + (1+\alpha_i+\cdots+\alpha_{i+n})a_i. \end{aligned}$$

The proof of the Theorem 1 is complete. ∎

Remark 1. If $\alpha_i = \alpha, \forall i$ we get Proposition 1 in [9].

Remark 2. From the proof of Theorem 1, it follows that

$$d(x_i, y_{i+n}) \leq (1 + \alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+n-1})d(x_i, y_i).$$

Indeed it is clear that

$$d(x_i, y_{i+n}) \leq d(x_i, y_i) + d(y_i, y_{i+1}) + \cdots + d(y_{i+n-1}, y_{i+n}).$$

Then by Property 2

$$\begin{aligned} d(x_i, y_{i+n}) &\leq d(x_i, y_i) + d(x_i, x_{i+1}) + \cdots + d(x_{i+n-1}, x_{i+n}) \\ &= d(x_i, y_i) + \alpha_i d(x_i, y_i) + \cdots + \alpha_{i+n-1}d(x_{i+n-1}, y_{i+n-1}), \end{aligned}$$

and finally using Property 3 we get the desired inequality.

3. Fixed Points of Non-Expansive Mapping

Theorem 2. *Let (X, d) be a metric space of hyperbolic type, $T : X \longrightarrow X$ a non-expansive mapping, and $\{x_n\}$ the sequence defined by (2) such that*

(i) $\inf_{n\in\mathbb{N}} \alpha_n = \alpha > 0, \sup_{n\in\mathbb{N}} \alpha_n = \beta < 1,$

(ii) *the sequence* $\{x_n\}$ *is bounded.*

Then

$$\lim_{n\longrightarrow\infty} d(x_n, y_n) = 0.$$

Proof. By hypotheses it is clear that there exists such a positive number A that

$$d(x_i, y_{i+n}) \leq A, \forall i, n \in \mathbb{N}.$$

Taking into account the decreasing sequence $\{d(x_n, y_n)\}$ and its boundedness by zero we can claim that there exists

$$\lim_{n\longrightarrow\infty} d(x_n, y_n) = r \geq 0.$$

It will be shown that $r = 0$.

Indeed, if $r > 0$, then $\forall\varepsilon > 0$, there exists a positive integer $N \geq \frac{A}{r\alpha}$, $\varepsilon(1-\beta)^{-N} < r$ and since the sequence $\{d(x_n, y_n)\}$ is Cauchy, for i large enough we obtain

$$d(x_i, y_i) - d(x_{i+N}, y_{i+N}) \leq \varepsilon.$$

Next we have

$$A + r \leq Nr\alpha + r = (N\alpha + 1)r \leq (1 + \alpha_i + \cdots + \alpha_{i+N-1})d(x_i, y_i).$$

Then by Theorem 1, for i large enough, we get

$$\begin{aligned} A + r &\leq d(x_i, y_{i+N}) + \frac{1}{(1-\alpha_i)\cdots(1-\alpha_{i+N-1})}[d(x_i, y_i) - d(x_{i+N}, y_{i+N})] \\ &\leq A + (1-\beta)^{-N}\varepsilon < A + r, \end{aligned}$$

a contradiction that proves $r = 0$.

Theorem 2 is thus proved. ■

As corollaries of Theorem 2, we obtain

Theorem 3. *Under the same assumptions as in Theorem 2, if the sequence* $\{x_n\}$ *has a subsequence converging to* $u \in X$, *then* u *is a fixed point of* T *and*

$$\lim_{n\longrightarrow\infty} x_n = u.$$

Theorem 4. *Under the same assumptions as in Theorem 2, if* $T(X)$ *lies in a compact subset of* X, *then the sequence* $\{x_n\}$ *defined by* (2) *converges to a fixed point of* T *for each* $x_0 \in X$.

Acknowledgement. The authors are grateful to the referee for his/her helpful remarks.

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