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Survey

A Survey of Associated and Coassociated Primes*

Siamak Yassemi

Department of Mathematics, University of Tehran
P.O. Box 13145-448, Tehran, Iran

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Abstract. In this article we will give a survey on some recent research on associated and coassociated primes. Some open questions and further directions of research are presented.

1. Associated Primes

The earliest impulse toward the development of what is now commutative algebra came from the desire of number theorists to make use of unique factorization in rings of integers in number fields other than $\mathbb{Q}$. When it became clear that unique factorization does not always hold, the search for the strongest available alternative began. The theory of primary decomposition is the direct result of that search.

Let R be a commutative ring and let M be a non-zero R-module. A non-zero submodule Q of M is called *primary* if, for each $a \in R$, the multiplication by a on M/Q is either injective or nilpotent. Then $\mathfrak{p} = \sqrt{(\mathrm{Ann}(M/Q)}$ is a prime ideal and Q is called $\mathfrak{p}$-*primary*. We say that M has a *primary decomposition* if there is a finite number of primary submodules $Q_1, Q_2, \ldots, Q_n$ such that $0 = Q_1 \cap Q_2 \cap \cdots \cap Q_n$. One may assume that the prime ideals $P_i = \sqrt{(\mathrm{Ann}(M/Q_i)}$, $i = 1, 2, \ldots, n$, are all distinct and, by omitting the redundant components, that the decomposition is minimal. Then the set of prime ideals $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ does not depend on the decomposition, and it is called the *set of associated prime ideals* and is denoted by $\mathrm{Ass}^*(M)$. If M is a Noetherian R-module, then M has

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a primary decomposition. Note that the set of R-modules that have primary decomposition strictly contains the set of Noetherian R-modules. (In [15] Iroze and Rush studied several notions of the associated primes of modules over a commutative ring.)

For a commutative Noetherian ring R, the set of *associated prime ideals* of M is denoted by $\mathrm{Ass}(M)$ and it is the set of prime ideals $\mathfrak{p}$ such that there exists $x \in M$ with the annihilator $\mathrm{Ann}(x)$ equal to $\mathfrak{p}$. In this case, it is easy to see that

$$\begin{aligned}\mathrm{Ass}_R(M) &= \{\mathfrak{p} \in \mathrm{Spec}(R) | \mathfrak{p} = \mathrm{Ann}(x) \text{ for some non-zero element } x \in M\}\\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) | \mathfrak{p} = \mathrm{Ann}(C) \text{ for some non-zero cyclic submodule } C \text{ of } M\}\\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) | \mathfrak{p} = \mathrm{Ann}(N) \text{ for some non-zero Noetherian submodule } N \text{ of } M\}\\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) | \mathfrak{p} = \mathrm{Z}(K) \text{ for some irreducible submodule } K \text{ of } M\}.\end{aligned}$$

Here K is *irreducible* if, for any submodules X and Y of K with $0 = X \cap Y$, either $X = 0$ or $Y = 0$. Also, $\mathrm{Z}(K)$ denotes the set of zero divisors of K. Let $\mathrm{Z}(M)$ the set of the elements $r \in R$ such that, the homothety $M \xrightarrow{r.} M$ is not injective, $\mathrm{nil}(M)$ be the set of elements $r \in R$ such that for each cyclic submodule N of M, there exists $n \in \mathbb{N}$ with $r^n N = 0$.

The basic properties of associated primes are collected in the next theorem.

Theorem 1.1. *Let R be a Noetherian ring. Then the following hold:*

(i) *If M is a finitely generated R-module, then* $\mathrm{Ass}(M) = \mathrm{Ass}^*(M)$;

(ii) $\mathrm{Ass}(M)$ *is non-empty if and only if M is non-zero;*

(iii) *If $0 \to K \to M \to L \to 0$ is a short exact sequence, then*

$$\mathrm{Ass}(K) \subseteq \mathrm{Ass}(M) \subseteq \mathrm{Ass}(K) \cup \mathrm{Ass}(L);$$

(iv) $\mathrm{Z}(M) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$;

(v) $\mathrm{nil}(M) = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$;

(vi) *If S is a multiplicative closed system of R, then*

$$\mathrm{Ass}_{S^{-1}R}(S^{-1}M) = \{\mathfrak{p}S^{-1}R \,|\, \mathfrak{p} \in \mathrm{Ass}(M) \textit{ with } \mathfrak{p} \cap S = \emptyset\}.$$

Now assume R is a commutative (not necessarily Noetherian) ring. For a Noetherian module M over R we can change the ring R to the ring $R/\mathrm{Ann}(M)$ which is Noetherian and prove all the properties in Theorem 1. Thus, for any Noetherian R-module M, we have $\mathrm{Ass}^*(M) = \mathrm{Ass}(M)$. Therefore, the new notion works for a larger set than the set of Noetherian R-modules.

For not necessarily Noetherian modules, the associated primes do not behave so well. For example, there exist non-trivial modules without associated primes (see Example 1). In [4] Bourbaki have introduced the notion of *weakly associated prime ideals* of M over R. A prime ideal $\mathfrak{p}$ is called weakly associated to M if there exists an element $x \in M$ such that $\mathfrak{p}$ is a prime ideal which is minimal among the prime ideals containing the annihilator $\mathrm{Ann}(x)$. The set of weakly associated primes of M is denoted by $\widetilde{\mathrm{Ass}}(M)$. In [4, p. 165–166], Bourbaki gave the following properties of weakly associated primes:

Theorem 1.2. *With notation as above, the following hold:*

(i) $\mathrm{Ass}(M) \subseteq \widetilde{\mathrm{Ass}}(M) \subseteq \mathrm{Supp}(M)$;

(ii) *For any decomposable module* M, *we have* $\widetilde{\mathrm{Ass}}_R(M) = \mathrm{Ass}^*_R(M)$ (cf. [36, Lemma 1.3]);

(iii) $\mathrm{Ass}(M) = \widetilde{\mathrm{Ass}}(M)$ *if* R *is a Noetherian ring, hence* $\mathrm{Ass}(M) = \widetilde{\mathrm{Ass}}(M)$ *if* M *is a Noetherian* R*-module* (cf. [36]);

(iv) $\widetilde{\mathrm{Ass}}(M)$ *is non-empty if and only if* M *is non-zero*;

(v) *If* $0 \to K \to M \to L$ *is a short exact sequence, then*

$$\widetilde{\mathrm{Ass}}(K) \subseteq \widetilde{\mathrm{Ass}}(M) \subseteq \widetilde{\mathrm{Ass}}(K) \cup \widetilde{\mathrm{Ass}}(L);$$

(vi) $\mathrm{Z}(M) = \bigcup_{\mathfrak{p} \in \widetilde{\mathrm{Ass}}(M)} \mathfrak{p}$;

(vii) $\mathrm{nil}(M) = \bigcap_{\mathfrak{p} \in \widetilde{\mathrm{Ass}}(M)} \mathfrak{p}$;

(viii) *If* S *is a multiplicative closed system of* R, *then*

$$\widetilde{\mathrm{Ass}}_{S^{-1}R}(S^{-1}M) = \{\mathfrak{p}S^{-1}R \mid \mathfrak{p} \in \widetilde{\mathrm{Ass}}(M) \text{ with } \mathfrak{p} \cap S = \emptyset\}.$$

The next example shows that the inclusion $\mathrm{Ass}(M) \subseteq \widetilde{\mathrm{Ass}}(M)$ can be strict.

Example 1. Let k be a field and consider the ring $R = k^N$ (direct product). Set $\mathfrak{a} = k^{(N)}$ (direct sum) which is an ideal of R. Set $M = R/\mathfrak{a}$. We claim that $\mathrm{Ass}(M)$ is empty. Assume that $\mathfrak{p} \in \mathrm{Ass}(M)$. Then $\mathfrak{a} \subseteq \mathfrak{p} = (\mathfrak{a} : r)$ for some $r \notin \mathfrak{a}$. It is easy to find two elements s and t of R such that $sr, tr \notin \mathfrak{a}$ and $st = 0$. Since $str = 0$ we have $st \in \mathfrak{p}$. But $s \notin \mathfrak{p}$ and $t \notin \mathfrak{p}$ and this is a contradiction.

It seems that the next result is the best one to date concerning the equality of weakly associated and associated primes.

Proposition 1.3. [37] *Let* M *be an* R*-module and let each weakly associated prime ideal of* M *be finitely generated. Then* $\mathrm{Ass}_R(M) = \widetilde{\mathrm{Ass}}_R(M)$.

The next example shows that there exist a non-Noetherian ring and a non-Noetherian module over it such that the set of associated primes and weakly associated primes of that module are equal.

Example 2. Let D be a domain that is non-Noetherian and let K be its field of quotients. Then $\widetilde{\mathrm{Ass}}_D(K) = \{0\}$. Thus $\mathrm{Ass}_D(K) = \widetilde{\mathrm{Ass}}_D(K)$ but D is a non-Noetherian ring and K is a non-Noetherian module.

2. Filtration

Let R be a Noetherian ring and let M be a finitely generated R-module. It is well known that there exists a chain $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of submodules of M together with prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for all $1 \leq i \leq n$, cf. [19, 6.4] and [14, 3.7]. It is also well known that the associated primes of M are among the primes $\mathfrak{p}_i$ appearing in the above filtration. In [14, p. 93], Eisenbud asked which modules M admit a filtration (as above) where, in addition, every $\mathfrak{p}_i$ is an associated prime of M. (In [17] Li answered this question in some special cases.) Such modules are called *clean*. It is noted in [14, p. 93] that when R is a domain and M is a torsion free R-module, M is clean if and only if M is free.

Now assume R is a (not necessarily Noetherian) commutative ring and M is an R-module. We say that the R-module M has a *weakly associated prime filtration* (WAPF for short) if there exists a chain $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of submodules of M together with prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n$ such that $\widetilde{\mathrm{Ass}}(M_i/M_{i-1}) = \{\mathfrak{p}_i\}$ for all $1 \leq i \leq n$. In this case we have $\widetilde{\mathrm{Ass}}(M) \subseteq \{\mathfrak{p}_1, \mathfrak{p}_2 \dots \mathfrak{p}_n\}$ by [35, 1.1]. We say that the R-module M has a *clean weakly associated prime filtration* (CWAPF for short) if there exists WAPF such that $\widetilde{\mathrm{Ass}}(M) = \{\mathfrak{p}_1, \mathfrak{p}_2 \dots \mathfrak{p}_n\}$.

The next example shows that CWAPF is not unique.

Example 3. Suppose $\mathfrak{p}$ is a prime ideal of R and $\mathrm{E}(R/\mathfrak{p})$ is the injective envelope of the domain $R/\mathfrak{p}$. By using Theorem 1.2 we have $\widetilde{\mathrm{Ass}}(\mathrm{E}(R/\mathfrak{p})) = \{\mathfrak{p}\}$. Therefore, $0 \subseteq \mathrm{E}(R/\mathfrak{p})$ is a CWAPF of $\mathrm{E}(R/\mathfrak{p})$. In addition, $0 \subseteq R/\mathfrak{p} \subseteq \mathrm{E}(R/\mathfrak{p})$ is a CWAPF.

Theorem 2.1. *Let M be an R-module. Then $|\widetilde{\mathrm{Ass}}(M)| < \infty$ if and only if the module M has a CWAPF.*

3. Hom and Tensor Functors

In [4] Bourbaki proved the following theorem.

Theorem 3.1. *Suppose R is a Noetherian ring. If M is a finitely generated R-module, then for any R-module N, we have*

$$\mathrm{Ass}_R(\mathrm{Hom}_R(M,N)) = \{\mathfrak{p} \in \mathrm{Ass}_R(N) | \mathfrak{q} \subseteq \mathfrak{p} \textit{ for some } \mathfrak{q} \in \mathrm{Ass}_R(M)\}.$$

The next example shows that the above theorem is not valid if we replace Ass by $\widetilde{\mathrm{Ass}}$.

Example 4. Let R and M be the same as in Example 1. Assume that $\mathfrak{m} \in \widetilde{\mathrm{Ass}}(M)$. Then $\mathfrak{m}$ belongs to the set $\mathrm{Max}(R)$ of maximal ideals of R since the ring R is von Neumann regular. Since $\mathfrak{m} \notin \mathrm{Ass}(M)$ we have $\mathrm{Hom}(R/\mathfrak{m}, M) = 0$.

Proposition 3.2. [36] *Let M and N be R-modules. If $\mathrm{Hom}_R(M,N) \neq 0$, then there exists $\mathfrak{p} \in \widetilde{\mathrm{Ass}}(M)$ such that $\mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in \widetilde{\mathrm{Ass}}_R(N)$.*

In special cases Theorem 3.1 is valid for non-Noetherian rings. For example, we have the following result.

Proposition 3.3. [10] *Let Q be a projective R-module and let N be a Noetherian R-module. Then*

$$\mathrm{Ass}_R(\mathrm{Hom}_R(Q,N)) = \{\mathfrak{p} \in \mathrm{Ass}_R(N) \mid \mathfrak{p} \supseteq \mathfrak{q} \text{ for some } \mathfrak{q} \in \widetilde{\mathrm{Ass}}_R(Q)\}.$$

4. Change of Rings

In this section let $\varphi : R \to S$ be a ring homomorphism and let M be a S-module.

Theorem 4.1. [19] *Let R and S be Noetherian rings. If M is a finitely generated S-module, then $\mathrm{Ass}_R(M) = \{\varphi^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Ass}_S(M)\}$.*

The following theorem is a generalization of Theorem 4.1.

Theorem 4.2. [37] *Let M be a S-module. Then*

$$\{\varphi^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Ass}_S(M)\} \subseteq \mathrm{Ass}_R(M) \subseteq \widetilde{\mathrm{Ass}}_R(M) \subseteq \{\varphi^{-1}(\mathfrak{p}) | \mathfrak{p} \in \widetilde{\mathrm{Ass}}_S(M)\}.$$

The inclusion in Theorem 4.2 may be strict (see [37, p. 2009]). It is clear that we have equalities in all steps if $\mathrm{Ass}_S(M) = \widetilde{\mathrm{Ass}}_S(M)$. For example, if S is a Noetherian ring or M is a Noetherian S-module, then we have equalities in all steps. In addition, if each of the elements in the set $\widetilde{\mathrm{Ass}}_S(M)$ are finitely generated ideal, then we have equalities in all steps.

The S-module M is called S-fine if $\widetilde{\mathrm{Ass}}_S(M) = \mathrm{Ass}_S(M)$. The next theorem is a generalization of Proposition 3.3.

Theorem 4.3. [37] *Let M be an S-fine module that is decomposable. Then for any projective R-module Q,*

$$\widetilde{\mathrm{Ass}}_S(\mathrm{Hom}_R(Q,M)) = \{\mathfrak{p} \in \widetilde{\mathrm{Ass}}_S(M) \mid \varphi^{-1}(\mathfrak{p}) \supseteq \mathfrak{q} \text{ for some } \mathfrak{q} \in \widetilde{\mathrm{Ass}}_R(Q)\}.$$

5. Asymptotic Associated Primes

In [5] Brodmann showed that if R is Noetherian and M is a finitely generated R-module, then the sets $\mathrm{Ass}_R(M/\mathfrak{a}M)$ and $\mathrm{Ass}(\mathfrak{a}^n M/\mathfrak{a}^{n+1}M)$ are independent of n for all large n. This was in response to a question of Ratliff who had shown in [28] that the sets $\mathrm{Ass}(R/\overline{\mathfrak{a}^n})$ are non-decreasing with n and constant for large n, where the bar denotes integral closure. Let $\mathrm{As}^{\#}(\mathfrak{a}, M)$ and $\mathrm{Bs}^{\#}(\mathfrak{a}, M)$ denote their ultimate constant values respectively. Then $\mathrm{As}^{\#}(\mathfrak{a}, M) - \mathrm{Bs}^{\#}(\mathfrak{a}, M) \subseteq \mathrm{Ass}(M)$ [21, Corollary 13]. There are several attempts to extend the above result. We list some of them.

Theorem 5.1. [16] *Let $I_1, I_2, \ldots, I_s$ be ideals of the ring R, M a Noetherian R-module, and N a submodule of M. If $(t_m(1), t_m(2), ..., t_m(s))_{m\in\mathbb{N}}$ is a sequence* *of s-tuples of non-negative integers which is non-decreasing in the sense that* $t_i(j) \leq t_{i+1}(j)$ *for all* $j = 1, 2, \ldots, s$ *and all* $i \in \mathbb{N}$, *then* $\mathrm{Ass}_R(M/I_1^{t_n(1)} \ldots I_s^{t_n(s)} N)$ *is independent of* n *for all large* n.

Theorem 5.2. [31] *If* M *is a Noetherian module and* N *is a submodule of* M, *then the sequence* $\mathrm{Ass}(\mathfrak{a}^n M/\mathfrak{a}^n N)$ *is constant for large* n.

Theorem 5.3. [23] *If* R *is Noetherian and* M *is a finitely generated* R*-module, then, for a given* $i \geq 0$, *the sequences of finite sets of associated primes* $\mathrm{Ass}(\mathrm{Tor}_i^R(R/\mathfrak{a}^n, M))$ *and* $\mathrm{Ass}(\mathrm{Tor}_i^R(\mathfrak{a}^n/\mathfrak{a}^{n+1}, M))$ *become independent of* n *for large* n.

Theorem 5.4. [10] *If* Q *is a projective* R*-module,* M *a Noetherian* R*-module and* N *a submodule of* M, *then the two sequences of associated primes* $\mathrm{Ass}(\mathrm{Hom}(Q, M)/\mathfrak{a}^n \mathrm{Hom}(Q, N))$ *and* $\mathrm{Ass}(\mathfrak{a}^n \mathrm{Hom}(Q, M)/\mathfrak{a}^n \mathrm{Hom}(Q, N))$ *become eventually constant.*

Proposition 5.5. [10] *If* R *is Noetherian,* Q *a projective* R*-module and* M *a finitely generated* R*-module then, for each* $i \geq 0$, *the two sequences of sets of associated primes* $\mathrm{Ass}(\mathrm{Tor}_i^R(R/\mathfrak{a}^n, \mathrm{Hom}(Q, M)))$ *and* $\mathrm{Ass}(\mathrm{Tor}_i^R(\mathfrak{a}^n/\mathfrak{a}^{n+1}, \mathrm{Hom}(Q, N)))$ *become eventually constant.*

(For the functorial generalizations of Theorem 5.4 and Proposition 5.5, see [13].)

Theorem 5.6. [1] *If* R *is Noetherian and* M *is a finitely generated* R*-module, then the sequences of sets* $\mathrm{Ass}(M \otimes N/\mathfrak{a}^n(M \otimes N))$ *and* $\mathrm{Ass}(\mathfrak{a}^n(M \otimes N)/\mathfrak{a}^{n+1}(M \otimes N))$ *are ultimately constant in the following cases:*

(i) N *is a finitely generated* R*-module.*

(ii) N *is an injective* R*-module.*

(iii) N *is an Artinian* R*-module.*

(iv) N *is a flat* R*-module.*

Proposition 5.7. [1] *Let* R *be a Noetherian ring and let* M *be a finitely generated* R*-module,* N *an Artinian* R*-module, and* F *a flat* R*-module. Then for a given* $i \geq 0$, *the sequences of the sets of associated primes* $\mathrm{Ass}(\mathrm{Tor}_i^R(R/\mathfrak{a}^n, N \otimes F))$ *and* $\mathrm{Ass}(\mathrm{Tor}_i^R(\mathfrak{a}^n/\mathfrak{a}^{n+1}, N \otimes F))$ *are ultimately constant.*

Let M be an R-module and let $\mathfrak{a}$ be a proper ideal of R. Set $\mathcal{R} = \mathrm{gr}_{\mathfrak{a}}(R) = \oplus_{n \geq 0} \mathfrak{a}^n/\mathfrak{a}^{n+1}$, and $\mathcal{M} = \mathrm{gr}_{\mathfrak{a}}(M) = \oplus_{n \geq 0} \mathfrak{a}^n M/\mathfrak{a}^{n+1} M$. Then $\mathcal{R}$ is a graded ring and $\mathcal{M}$ is a graded $\mathcal{R}$-module. Let $\varphi : R \to \mathcal{R}$ denote the composition of the natural homomorphisms $R \to R/\mathfrak{a} \to \mathcal{R}$.

Theorem 5.8. [39] *If the set* $\widetilde{\mathrm{Ass}}_{\mathcal{R}}(\mathcal{M})$ *is finite, then the set* $\bigcup_{n \geq 0} \widetilde{\mathrm{Ass}}_R(M/\mathfrak{a}^{n+1} M)$ *is finite.*

We have

$$\widetilde{\mathrm{Ass}}_R(\mathfrak{a}^n M/\mathfrak{a}^{n+1}M) \subseteq \widetilde{\mathrm{Ass}}_R(\mathcal{M}) \subseteq \{\varphi^{-1}(\mathfrak{p}) | \mathfrak{p} \in \widetilde{\mathrm{Ass}}_{\mathcal{R}}(M)\}.$$

Question 5.9. Is the sequence of sets $\widetilde{\mathrm{Ass}}_R(M/\mathfrak{a}^{n+1}M)$ independent of n for n large enough if the set $\widetilde{\mathrm{Ass}}_{\mathcal{R}}(\mathcal{M})$ is finite?

6. Attached Primes

There have been several attempts to dualize the theory of associated primes. The first one was made by Macdonald in [18] by defining the set of attached prime ideals and secondary representation of a module, which is (in certain sense) a dual to the theory of associated prime ideals and primary decomposition. A non-zero R-module M is called *secondary* if, for each $a \in R$, multiplication by a on M is either surjective or nilpotent. Then $\mathrm{nil}(M) = \mathfrak{p}$ is a prime ideal and M is called $\mathfrak{p}$-*secondary.* We say that M has a secondary representation (representable) if there is a finite number of secondary submodules $S_1, S_2, \ldots, S_n$ such that $M = S_1 + S_2 + \cdots + S_n$. One may assume that the prime ideals $\mathrm{nil}(S_i) = \mathfrak{p}_i, i = 1, 2, \ldots, n$, are all distinct and, by omitting the redundant summands, that the representation is minimal. Then the set of prime ideals $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ does not depend on the representation, and it is called the set of attached prime ideals and is denoted by $\mathrm{Coass}^*(M)$ (in [18] it is denoted by $\mathrm{Att}(M)$).

In [18] Macdonald showed that every Artinian R-module is representable. But the set of representable R-modules strictly contains the set of Artinian R-modules. For example, the injective modules over Noetherian ring are representable (cf. [32]). This result was extended by Ansari and Sharp.

Theorem 6.1. [2, Theorem 2.1] *Let R be a Noetherian ring, M a finitely generated R-module and E an injective R-module. Then* $\mathrm{Hom}_R(M, E)$ *is a representable R-module.*

In [22] Melkersson and Schenzel succeeded to drop the Noetherian condition for R:

Theorem 6.2. [22] *Let R be a ring, M a Noetherian R-module, and E an injective R-module. Then* $\mathrm{Hom}_R(M, E)$ *is a representable R-module.*

In addition there are other attempts to find more representable modules. For example, we have the following theorems.

Theorem 6.3. [26] *If R is a Noetherian ring, M an Artinian, and F a flat R-modules, then the module* $\mathrm{Hom}(F, M)$ *is representable.*

Question 6.4. Let R be a Noetherian ring, M a representable R-module, and F a flat R-module. Is the module $\mathrm{Hom}(F, M)$ representable?

The attached primes is particularly well-behaved when M has a secondary representation. However, in general this theory is not completely satisfactory.

7. Coassociated Primes

The second attempt to dualizing the theory of associated primes was made by Chambless in [6] by defining the set of coassociated prime ideals of an R-module M to be the set of prime ideals $\mathfrak{p}$ such that there exists a sum-irreducible homomorphic image L of M with $\mathfrak{p}$ equal to the set $\{a \in R|aL \neq L\}$. (Here N is said to be sum-irreducible if, for any submodules X and Y with $N = X + Y$, either $X = N$ or $Y = N$.)

The next attempt was made by Zöshinger in [40, 41]. He defined coassociated primes of the R-module M to be the set of prime ideals such that there exists an Artinian homomorphic image L of M with $\mathfrak{p} = \mathrm{Ann}(L)$. He noted that this definition is equivalent to Chambless' definition, and gave some related fundamental results about this concept.

Finally, in [35] the present author introduced the concept of cocyclic modules which is a dual to cyclic modules, and we used it to define coassociated prime ideals of modules over Noetherian rings. Then in [36] we extended this notion for modules over commutative (not necessarily Noetherian) rings.

Definition 7.1. *An R-module L is said to be cocyclic if L is isomorphic to a submodule of $\mathrm{E}(R/\mathfrak{m})$ for some $\mathfrak{m} \in \mathrm{Max}(R)$.*

Remark 1. If $(R, \mathfrak{m})$ is a complete local Noetherian ring then M is cocyclic if and only if there exists an ideal $\mathfrak{a}$ of R such that $M \cong \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{E}(R/\mathfrak{m}))$ (cf. [35, Remark after (1.2)]).

It is well known that every finitely generated module is a homomorphic image of a finite direct sum of cyclic submodules of itself. An R-module M is said to be finitely cogenerated (the dual notion of finitely generated) if $\mathrm{E}(M)$ is isomorphic to a direct sum of finitely many injective envelope of simple modules. Now we bring a dual of this result.

Theorem 7.2. [38] *Every finitely cogenerated module can be embedded in a finite direct sum of cocyclic homomorphic images of itself.*

Theorem 7.3. [38] *Every cocyclic R-module is Artinian (resp. Noetherian) if and only if $R_{\mathfrak{m}}$ is Noetherian (resp. Artinian) for every $\mathfrak{m} \in \mathrm{Max}(R)$.*

There is a characterization of von Neumann regular rings with using cocyclic modules.

Theorem 7.4. [38] *The ring R is von Neumann regular if and only if each cocyclic R-module is simple.*

Definition 7.5. *Let M be an R-module. A prime ideal $\mathfrak{p}$ of R is called a coassociated (resp. weakly coassociated) prime of M if there exists a cocyclic homomorphic image L of M such that $\mathfrak{p} = \mathrm{Ann}(L)$ (resp. $\mathfrak{p} \in \mathrm{Min}(\mathrm{Ann}(L))$, the minimal prime ideals contain $\mathrm{Ann}(L)$. The set of coassociated (resp. weakly coassociated) prime ideals of M is denoted by $\mathrm{Coass}(M)$ (resp. $\widetilde{\mathrm{Coass}}(M)$).*

Definition 7.6. *Let M be an R-module. The cosupport of M, written as $\mathrm{Cosupp}(M)$, is the set of prime ideals $\mathfrak{p}$ such that there exists a cocyclic homomorphic image L of M with $\mathfrak{p} \supseteq \mathrm{Ann}(L)$.*

The following theorem is an important tool in the study of coassociated prime ideals.

Theorem 7.7. [36] *Let M be an R-module. The following are equivalent:*
(i) $\mathfrak{p} \in \widetilde{\mathrm{Coass}}(M)$;
(ii) *There exists* $\mathfrak{m} \in \mathrm{Max}(R)$ *containing* $\mathfrak{p}$ *such that* $\mathfrak{p} \in \widetilde{\mathrm{Ass}}(\mathrm{Hom}_R(M, \mathrm{E}(R/\mathfrak{m})))$.

Theorem 7.8. [35] *Consider the following statements:*
(i) $\mathfrak{p} \in \widetilde{\mathrm{Ass}}(M)$;
(ii) *There exists* $\mathfrak{m} \in \mathrm{Max}(R)$ *containing* $\mathfrak{p}$ *such that* $\mathfrak{p} \in \widetilde{\mathrm{Coass}}(\mathrm{Hom}_R(M, \mathrm{E}(R/\mathfrak{m})))$. *For any R-module M,* (i) *implies* (ii) *and if M is Noetherian, then* (ii) *implies* (i).

Denote by $\mathrm{W}(M)$ the set of the elements $r \in R$ such that the homothety $M \xrightarrow{r\cdot} M$ is not surjective; $\mathrm{Conil}(M)$ the set of all elements $r \in R$ such that, for each cocyclic homomorphic image L of M, there exists $n \in \mathbb{N}$ with $r^n L = 0$.

Theorem 7.9. [36] *Let M be an R-module. The following hold*
(a) $\mathrm{Coass}(M) \subseteq \widetilde{\mathrm{Coass}}(M) \subseteq \mathrm{Cosupp}(M)$;
(b) *If M is representable, then* $\widetilde{\mathrm{Coass}}(M) = \mathrm{Coass}^*(M)$;
(c) *If R is a Noetherian ring, then* $\widetilde{\mathrm{Coass}}(M) = \mathrm{Coass}(M)$;
(d) *If M is an Artinian R-module, then* $\widetilde{\mathrm{Coass}}(M) = \mathrm{Coass}(M)$;
(e) *If* $0 \to K \to M \to L$ *is a short exact sequence, then*

$$\widetilde{\mathrm{Ccoass}}(L) \subseteq \widetilde{\mathrm{Coass}}(M) \subseteq \widetilde{\mathrm{Coass}}(K)\widetilde{\mathrm{Coass}}(L);$$

(f) $\mathrm{W}(M) = \cup_{\mathfrak{p} \in \widetilde{\mathrm{Coass}}(M)} \mathfrak{p}$;
(g) $\mathrm{Conil}(M) = \cap_{\mathfrak{p} \in \widetilde{\mathrm{Coass}}(M)} \mathfrak{p}$;
(h) *If M is representable, then $\widetilde{\mathrm{Coass}}(M)$ has finitely many elements. In particular, if M is an Artinian R-module, then $\widetilde{\mathrm{Coass}}(M)$ is a finite set;*
(i) *If M is representable, then* $\widetilde{\mathrm{Ass}}(R/\mathrm{Ann}(M)) \subseteq \widetilde{\mathrm{Coass}}(M)$ *and the sets* $\mathrm{V}(\mathrm{Ann}(M))$, $\widetilde{\mathrm{Ass}}(R/\mathrm{Ann}(M))$, *and* $\widetilde{\mathrm{Coass}}(M)$ *have the same minimal elements.*

8. Coassociated Primes with Hom and Tensor Functors

First we bring the dual of Theorem 3.1.

Theorem 8.1. [35] *Suppose R is a Noetherian ring. If M is a finitely generated R-module, then for any R-module N, we have*

$$\mathrm{Coass}(M \otimes N) = \{\mathfrak{p} \in \mathrm{Coass}_R(N) \mid \mathfrak{q} \subseteq \mathfrak{p} \textit{ for some } \mathfrak{q} \in \mathrm{Ass}_R(M)\}.$$

In the following theorems we will specify the set of attached primes for some representable modules.

Theorem 8.2. [2] *Let R be Noetherian and let M be a finitely generated R-module. If E is an injective R-module, then*

$$\mathrm{Att}(\mathrm{Hom}(M, E)) = \{\mathfrak{p} \in \mathrm{Ass}(M) \mid \mathfrak{p} \subseteq \mathfrak{q} \textit{ for some } \mathfrak{q} \in \mathrm{Ass}(E)\}.$$

Theorem 8.3. [22] *Let R be a ring and let M be a Noetherian R-module. If E is an injective R-module then*

$$\mathrm{Att}(\mathrm{Hom}(M, E)) = \{\mathfrak{p} \in \mathrm{Ass}(M) \mid \mathfrak{p} \subseteq \mathfrak{q} \textit{ for some } \mathfrak{q} \in \mathrm{Ass}(E)\}.$$

Remark 2. Let M be an R-module and let E be an injective R module. Then

$$\widetilde{\mathrm{Coass}}(\mathrm{Hom}(M, E)) \supseteq \{\mathfrak{p} \in \widetilde{\mathrm{Ass}}(M) \mid \mathfrak{p} \subseteq \mathfrak{q} \text{ for some } \mathfrak{q} \in \widetilde{\mathrm{Ass}}(E)\},$$

and the equality does not hold in general (cf. [35, Example after (1.8)]).

Lemma 8.4. [10] *Let Q be a projective R-module, M a Noetherian and N an Artinian R-modules. Then*

$$\mathrm{Att}(\mathrm{Hom}(Q, N)) = \{\mathfrak{p} \in \mathrm{Att}(N) \mid \mathfrak{p} \supseteq \mathfrak{q} \textit{ for some } \mathfrak{q} \in \widetilde{\mathrm{Ass}}(Q)\}.$$

Theorem 8.5. [1] *Let R be a complete local ring. Let F be a flat and N an Artinian R-modules. Then*

$$\mathrm{Att}(\mathrm{Hom}(F, N)) = \{\mathfrak{p} \in \mathrm{Att}(N) \mid \mathfrak{p} \subseteq \mathfrak{q} \textit{ for some } \mathfrak{q} \in \widetilde{\mathrm{Coass}}(F)\}.$$

The above theorem is still valid if we replace the condition "complete local ring" by "Noetherian ring" (see [11, 3.3]).

Question 8.6. Is the above theorem valid for arbitrary commutative ring R?

9. Coassociated Primes Under Change of Rings

Let $\phi : R \to S$ be a homomorphism of rings. Let M be a S-module. There is no general result which determines whether weakly associated primes of S-module M necessarily contract to weakly coassociated primes of M as an R-module. However there are some results in special cases.

Proposition 9.1. [25] *Let the S-module M have secondary representation. Then*

$$\widehat{\mathrm{Coass}}_R(M) = \{\phi^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in \widehat{\mathrm{Coass}}_S(M)\}.$$

In general we do not have the equality in the above proposition, as the next example shows.

Example 5. [12] Let R be a Noetherian integral domain and let $\mathfrak{p}$ be a prime ideal such that $\mathfrak{p} \notin \mathrm{Max}(R) \cup \mathrm{Min}(R)$. Let $\phi : R \to R_{\mathfrak{p}}$ be the natural map. Then $\widehat{\mathrm{Coass}}_R(R_{\mathfrak{p}}) \neq \{\phi^{-1}(\mathfrak{p}) \mid \mathfrak{p} \in \widehat{\mathrm{Coass}}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})\}$.

Theorem 9.2. [37] *Let M be an R-module and let $\mathfrak{p}$ be a prime ideal of R. Then*

$$\widehat{\mathrm{Coass}}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \{\mathfrak{q}R_{\mathfrak{p}} \mid \mathfrak{q} \in \widehat{\mathrm{Ass}}_R(\mathrm{Hom}_R(M, \mathrm{E}(R/\mathfrak{p})))\}.$$

10. Asymptotic Attached Primes

Sharp in [33] has shown that, for an Artinian module M, the sequence of the sets $\mathrm{Att}(0 :_M \mathfrak{a}^n)$ and $\mathrm{Att}(0 :_M \mathfrak{a}^{n+1}/0 :_M \mathfrak{a}^n)$ are ultimately constant for large $n \in \mathbb{N}$ and if $\mathrm{At}^{\#}(\mathfrak{a}, M)$ and $\mathrm{Bt}^{\#}(\mathfrak{a}, M)$ denote, respectively, their ultimate constant values, then $\mathrm{At}^{\#}(\mathfrak{a}, M) - \mathrm{Bt}^{\#}(\mathfrak{a}, M) \subseteq \mathrm{Att}(M)$. There are several attempts to extend this result. We list some of them.

Theorem 10.1. [16] *Let $I_1, I_2, \ldots, I_s$ be ideals of the ring R, let M be an Artinian R-module and let N be a submodule of M. If $(t_m(1), t_m(2), \ldots, t_m(s))_{m\in\mathbb{N}}$ is a sequence of s-tuples of non-negative integers which is non-decreasing in the sense that $t_i(j) \leq t_{i+1}(j)$ for all $j = 1, 2, \ldots, s$ and all $i \in \mathbb{N}$, then $\mathrm{Att}_R(N :_M I_1^{t_n(1)} \ldots I_s^{t_n(s)})$ is independent of n for all large n.*

Theorem 10.2. [31] *If M is Artinian and $N' \subseteq N$ are submodules of M, then the sequence $\mathrm{Att}(N :_M \mathfrak{a}^n / N' :_M \mathfrak{a}^n)$ is constant for large n. In addition, $\mathrm{Att}(N :_M \mathfrak{a}^n)$ is constant for large n.*

Theorem 10.3. [23] *Let R be Noetherian and let M be an Artinian R-module. For a given $i \geq 0$ the sequences of finite sets of attached prime ideals $\mathrm{Att}(\mathrm{Ext}^i(R/\mathfrak{a}^n, M))$ and $\mathrm{Att}(\mathrm{Ext}^i(\mathfrak{a}^n/\mathfrak{a}^{n+1}, M))$, $n \in \mathbb{N}$, become for large n independent of n.*

Theorem 10.4. [10] *Let Q be a projective R-module. Let M be an Artinian R-module and let $N' \subseteq N$ be submodules of M. Then the sequences*

(i) $\mathrm{Att}(\mathrm{Hom}(Q,N) :_{\mathrm{Hom}(Q,M)} \mathfrak{a}^n)$, $n \in \mathbb{N}$,

(ii) $\mathrm{Att}(\mathrm{Hom}(Q,N) :_{\mathrm{Hom}(Q,M)} \mathfrak{a}^n / (\mathrm{Hom}(Q,N') :_{\mathrm{Hom}(Q,M)} \mathfrak{a}^n))$, $n \in \mathbb{N}$,

become eventually constant. In particular the sets of attached primes $\mathrm{Att}(0 :_{\mathrm{Hom}(Q,M)} \mathfrak{a}^n)$ *and* $\mathrm{Att}(0 :_{\mathrm{Hom}(Q,M)} \mathfrak{a}^{n+1} / 0 :_{\mathrm{Hom}(Q,M)} \mathfrak{a}^n)$, $n \in \mathbb{N}$, *become eventually constant and if* $\mathrm{At}^{\#}(\mathfrak{a}, \mathrm{Hom}(Q,M))$ *and* $\mathrm{Bt}^{\#}(\mathfrak{a}, \mathrm{Hom}(Q,M))$ *denote, respectively, their ultimate constant values, then* $\mathrm{At}^{\#}(\mathfrak{a}, \mathrm{Hom}(Q,M)) - \mathrm{Bt}^{\#}(\mathfrak{a}, \mathrm{Hom}(Q,M)) \subseteq \mathrm{Att}(M) \cap \mathrm{Supp}(Q)$.

Theorem 10.5. [10] *Let R be Noetherian, Q a projective and M an Artinian R-modules. Then the sequences of sets*

(i) $\mathrm{Att}(\mathrm{Ext}^i(R/\mathfrak{a}^n, \mathrm{Hom}(Q,M)))$, $n \in \mathbb{N}$,

(ii) $\mathrm{Att}(\mathrm{Ext}^i(\mathfrak{a}^n/\mathfrak{a}^{n+1}, \mathrm{Hom}(Q,M)))$, $n \in \mathbb{N}$,

become eventually constant.

There is a functorial generalization of Theorems 10.4 and 10.5 (see [13]).

Theorem 10.6. [1] *Let N, M, and E be respectively a finitely generated, an Artinian and an injective R-module. Then for a given $i \geq 0$, the sequences of the sets*

(i) $\mathrm{Att}(\mathrm{Ext}^i(N/\mathfrak{a}^n N, M))$, $n \in \mathbb{N}$,

(i') $\mathrm{Att}(\mathrm{Ext}^i(\mathfrak{a}^n N/\mathfrak{a}^{n+1} N, M))$, $n \in \mathbb{N}$,

(ii) $\mathrm{Coass}(\mathrm{Ext}^i(R/\mathfrak{a}^n, \mathrm{Hom}(N,E)))$, $n \in \mathbb{N}$,

(ii') $\mathrm{Coass}(\mathrm{Ext}^i(\mathfrak{a}^n/\mathfrak{a}^{n+1}, \mathrm{Hom}(N,E)))$, $n \in \mathbb{N}$,

are ultimately constant.

Corollary 10.7. [1] *Let R be a Noetherian ring, M an Artinian and F a flat R-modules. Let $N' \subseteq N$ be submodules of M. Then the sequences of the sets of attached primes*

(i) $\mathrm{Att}(\mathrm{Hom}(F,N) :_{\mathrm{Hom}(F,M)} \mathfrak{a}^n)$, $n \in \mathbb{N}$,

(ii) $\mathrm{Att}(\mathrm{Hom}(F,N) :_{\mathrm{Hom}(F,M)} \mathfrak{a}^n)/(\mathrm{Hom}(F,N') :_{\mathrm{Hom}(F,M)} \mathfrak{a}^n)$, $n \in \mathbb{N}$,

are ultimately constant.

11. Colocalization

It is well known that the prime ideal $\mathfrak{p}$ belongs to $\mathrm{Supp}(M)$ if and only if the localization $M_{\mathfrak{p}}$ is non-zero. It is natural to believe that the dual of this result is: "The prime ideal $\mathfrak{p}$ belongs to $\mathrm{Cosupp}(M)$ if and only if $\mathrm{Hom}_R(R_{\mathfrak{p}}, M)$ is non-zero." In fact "if" does hold (see [35, 2.16]), while "only if" holds for Artinian modules (see [24, 7.3]) and for injective modules (see [35, 2.18]) but it does not hold in general (see [24, p. 9]).

Question 11.1. Let M have a secondary representation. Is it true that the prime ideal $\mathfrak{p}$ belongs to $\mathrm{Cosupp}(M)$ if and only if $\mathrm{Hom}_R(R_\mathfrak{p}, M)$ is non-zero?

On the other hand, Smith in [22, p. 23] noted that, for a local ring $(R, \mathfrak{m})$, the functors $\mathrm{Hom}(\mathrm{Hom}(-, \mathrm{E}(R/\mathfrak{m})), \mathrm{E}(R/\mathfrak{p}))$ for $\mathfrak{p} \in \mathrm{Spec}(R)$ have properties dual to localization. The next theorem is another kind of colocalization.

Theorem 11.2. [35] *Let M be an R-module. Then $\mathfrak{p} \in \mathrm{Cosupp}(M)$ if and only if* $\mathrm{Hom}(\coprod_{\mathfrak{m}\in \mathrm{Max}(R)} \mathrm{Hom}(M, \mathrm{E}(R/\mathfrak{m})), \mathrm{E}(R/\mathfrak{p})) \neq 0$.

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References

1. H. Ansari-Toroghy, Associated and coassociated primes, *Commun. Algebra* **26** (1998) 453-466.
2. H. Ansari-Toroghy and R. Y. Sharp, Asymptotic behaviour of ideals relative to injective modules over commutative Noetherian rings, *Proc. Edinburgh Math. Soc.* **34** (1991) 155-160.
3. R. W. Berger, Various notions of associated prime ideals, *Annal. Univer. Saraviensis* **5** (1994) 245-271.
4. N. Bourbaki, *Algèbre Commutative*, Herman, Paris, 1967, Chapters 3 and 4.
5. M. Brodmann, Asymptotic stability of $\mathrm{Ass}(M/I^n M)$, *Proc. Amer. Math. Soc.* **74** (1979) 16-18.
6. L. Chambless, Coprimary decomposition, N-dimension and divisibility, Application to Artinian modules, *Commun. Algebra* **9** (1981) 1131-1146.
7. S.-C. Chung, Associated prime ideals and isomorphisms of modules of generalized fractions, *Math. J. Toyama University* **17** (1994) 175-205.
8. S.-C. Chung, Triangular subsets and coassociated prime ideals, *J. Chungcheong Math. Soc.* **9** (1996) 17-25.
9. P. Dutton, Prime ideals attached to a module, *Quart. J. Math. Oxford* **29** (1978) 403-413.
10. K. Divaani-Aazar and M. Tousi, Asymptotic associated and attached prime ideals related to projective modules, *Commun. Algebra* **25** (1997) 1129-1142.
11. K. Divaani-Aazar and M. Tousi, Asymptotic attached primes related to flat modules, *Algebra Colloq.* (to appear).
12. K. Divaani-Aazar and M. Tousi, Some remarks on coassociated primes, *J. Korean Math. Soc.* (to appear).
13. K. Divaani-Aazar and M. Tousi, Asymptotic prime ideals related to exact functors, *Commun. Algebra* **27** (1999) 3949-3968.
14. D. Eisenbud, *Commutative Algebra*, Springer-Verlag, 1995.
15. J. Iroz and D. E. Rush, Associated prime ideals in non-Noetherian rings, *Canad. J. Math.* **36** (1984) 344-360.
16. A. K. Kingsbury and R. Y. Sharp, Asymptotic behaviour of certain sets of prime ideals, *Proc. Amer. Math. Soc.* **124** (1996) 1703-1711.

17. A. Li, Associated prime filtrations of finitely generated modules over Noetherian rings, *Commun. Algebra* **23** (4) (1995) 1511-1526.
18. I. G. Macdonald, Secondary representation of modules over a commutative ring, *Sympos. Math.* **XI** (1973) 23-43.
19. H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, Cambridge, 1986.
20. S. McAdam, *Asymptotic Prime Divisors*, Lecture Notes in Mathematics, Vol. 1023, Springer-Verlag, 1983.
21. S. McAdam and P. Eakin, The asymptotic Ass, *J. Algebra* **61** (1979) 71-81.
22. L. Melkersson and P. Schenzel, Asymptotic attached prime ideals related to injective modules, *Commun. Algebra* **20** (1992) 583-590.
23. L. Melkersson and P. Schenzel, Asymptotic prime ideals related to derived functors, *Proc. Amer. Math. Soc.* **117** (1993) 935-938.
24. L. Melkersson and P. Schenzel, The co-localization of an Artinian module, *Proc. Edinburgh Math. Soc.* **38** (1995) 121-131.
25. L. Melkersson, On asymptotic stability for sets of prime ideals connected with the powers of an ideal, *Math. Proc. Cambridge Philos. Soc.* **107** (1990) 267-271.
26. L. Melkersson, Cohomological properties of modules with secondary representations, *Math. Scand.* **77** (1995) 197-208.
27. I. Nishitani, Associated prime ideals of the dual of an Artinian module relative to an injective module, *Commun. Algebra* **22** (1994) 2651-2667.
28. D. G. Northcott, Remarks on the theory of attached prime ideals, *Quart. J. Math. Oxford* **33** (1982) 239-245.
29. L. J. Ratliff Jr., On prime divisors of I^n, n large, *Michigan Math. J.* **23** (1976) 337-352.
30. L. J. Ratliff Jr., On asymptotic prime divisors, *Pacific J. Math.* **111** (1984) 395-413.
31. D. E. Rush, Asymptotic primes and integral closure in modules, *Quart. J. Math. Oxford* **43** (1992) 477-499.
32. R. Y. Sharp, Secondary representations for injective modules over commutative Noetherian rings, *Proc. Edinburgh Math. Soc.* **20** (1976) 143-151.
33. R. Y. Sharp, Asymptotic behavior of certain sets of attached prime ideals, *J. London Math. Soc.* **34** (1986) 212-218.
34. K. E. Smith, Tight closure of parameter ideals and F-rationality, Ph. D. Thesis, University of Michigan, 1993.
35. S. Yassemi, Coassociated primes, *Commun. Algebra* **23** (1995) 1473-1498.
36. S. Yassemi, Coassociated primes of modules over commutative rings, *Math. Scand.* **80** (1997) 175-187.
37. S. Yassemi, Weakly associated primes under change of rings, *Commun. Algebra* **26** (1998) 2007-2018.
38. S. Yassemi, Dual notion of cyclic modules, *Kobe J. Math.* **15** (1998) 41-46.
39. S. Yassemi, Weakly associated prime filtration, *Acta Math. Hungarica* (to appear).
40. H. Zöschinger, Über koassoziierte Primideale, *Math. Scand.* **63** (1988) 196-211.
41. H. Zöschinger, Über koassoziierte Primideale, II, *Math. Scand.* **81** (1997) 247-259.