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An Algorithm for Optimizing Over the Efficient Set*

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Abstract. In this paper we consider the problem of optimizing a linear function over the efficient set of a multiobjective linear programming problem. Basing on Philip's approach and using normal cone method for finding efficient edges and vertices adjacent to a given efficient vertex, we present an algorithm for solving this problem. Some illustrative examples are given.

1. Introduction

Let M be a nonempty polyhedral convex set in R^n determined by a system of linear inequations

$$\langle a^i, x\rangle \geq b_i, i = 1, \dots, m, \tag{1}$$

where $a^i \in R^n$ and $b_i \in R$, $i = 1, \dots, m$. Let C be a $(p \times n)$-real matrix with p rows $c^i \in R^n$. Consider the problem

$$\text{(P)} \qquad \min\langle d, x\rangle, \text{ subject to } x \in E_M,$$

where $d \in R^n$ and E_M is the efficient solution set of the multiobjective linear programming problem

$$\text{(VP)} \qquad \operatorname{Min} Cx, \text{ subject to } x \in M.$$

It is well-known that the efficient set E_M is a connected set and, in general, it is a complicated nonconvex subset of the boundary of the polyhedron M. Problem (P) is one of nonconvex programming problems in which any local solution may not be a global one. Since E_M is the union of faces of M, Problem

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(P) attains its global optimal solution at a vertex of M.

Although not nearly as extensively studied as Problem (VP), Problem (P) has received increasing attention in recent years. Many algorithmic ideas and algorithms for solving this problem have been proposed; see, for example, Benson [3 - 5], Bolintineanu [6], Ecker and Song[8], Isermann and Steuer [11], Fülöp [10], An, Muu and Tao [1], Luc and Muu [14], Muu [15], Philip [16], Steuer [19], Thach [20], Yu [21] and references therein.

In [16], Philip first studied Problem (P) and schematically described a cutting plane procedure for solving it. Later, Isermann and Steuer in [11] proposed a similar procedure for solving (P) where the objective function $\langle d, x\rangle$ is one of the multiple objectives $\langle c^i, x\rangle$ in (VP). In [8], Ecker and Song used Philip's approach presented two implementable algorithms that involve a privoting technique on the feasible set for (VP) or a reduced feasible set. Recently Philip's method was implemented by Bolintineanu [6] for the case where the objective function of (P) is quasiconcave. Fülöp in [10] formulated Problem (P) as a linear program with an additional reserve convex constraint and proposed a cutting plane method using facet cuts for solving the latter problem.

Basing on the study of normal cones and their relationship with efficient solution faces for (VP), Kim and Luc [12] have proposed a quite simple method for generating the whole efficient set for this problem, which takes into account the degenerate case. In this paper, we present an algorithm for solving the problem (P). It enjoys advantages of both Philip's approach [16] and the technique proposed in [12] for finding efficient edges and vertices adjacent to a given efficient vertex.

In Sec. 2 some descriptions for efficiency to Problem (VP) in terms of negative normal cone are presented. Sec. 3 is concerned with the cutting plane method and the reduced Problem (RVP). The results obtained in this section play the basic role for the algorithm described in Sec. 4. Some computational examples are given in the last section.

Throughout this paper, $M \subset R^n$ is the polyhedral set determined by the system (1) and C is a fixed matrix of the objective functions c^i, $i = 1, \dots, p$. Furthermore, without loss of generality, we always assume that there is no redundant inequality in (1) and the interior of M is not empty. For two vectors $z^1, z^2 \in R^p$, $z^i = (z_1^i, \dots, z_p^i)$, we write

$z^1 \geq z^2$ if and only if $z_i^1 \geq z_i^2$ for all $i = 1, \dots, p$;

$z^1 > z^2$ if and only if $z^1 \geq z^2$ and $z^1 \neq z^2$.

2. Efficiency and Negative Normal Index Sets

In this section we consider Problem (VP) formulated as in the previous section. Recall that the efficient set E_M for (VP) is the set of all points $x^0 \in M$ such that there is no other $x \in M$ such that $Cx^0 > Cx$. The normal cone [17] to a convex set $X \subset R^n$ at a point $x^0 \in X$, denoted by $N_X(x^0)$, consists of the outward normals to the supporting half-spaces to X at x^0, i.e.

$$N_X(x^0) = \{v \in R^n : \langle v, x - x^0\rangle \leq 0 \text{ for all } x \in X\}.$$

When M is determined by (1), the normal cone to M at a point $x^0 \in M$ can be represented as follows.

Proposition 2.1 [18, Theorem 6.46]. *Let $x^0 \in M$ satisfy the following equations and inequations*

$$\langle a^i, x\rangle = b_i, i \in I(x^0)$$
$$\langle a^j, x\rangle > b_j,\ j \in \{1, \ldots, m\}\backslash I(x^0),$$

where $I(x^0)$ is a nonempty index subset of $\{1, \ldots, m\}$. Then

$$N_M(x^0) = \text{cone}\{-a^i, i \in I(x^0)\}.$$

For convenience, let us recall some relations between the normal cones to M and the efficiency of Problem (VP) established in [12], which will be used in the further comming sections. A vector $v \in R^n$ is said to be *C-positive* if there exist strictly positive numbers $\lambda_1, \ldots, \lambda_p$ such that $v = \sum_{i=1}^{p} \lambda_i c^i$. If $-v$ is C-positive, then v is called *C-negative.* We say that the normal cone to M at $x^0 \in M$ is *negative* if it contains a C-negative vector.

The following proposition provides another description for the efficient solutions for (VP) in terms of negative normal cones.

Proposition 2.2 [12, Proposition 5.2]. *A point $x^0 \in M$ is an efficient solution for* (VP) *if and only if the normal cone to M at x^0 is negative, i.e. $N_M(x^0)$ contains a C-negative vector.*

It may be more useful in computation if instead of normal cones we work with the index set of the vectors generating them as in Proposition 2.1. Following [12], an index set $I \subseteq \{1, ..., m\}$ is said to be a *normal set* if there is some point $x^0 \in M$ such that the normal cone to M at x^0 coincides with the cone generated by $\{-a^i : i \in I\}$. It is obvious that not every subset of $\{1, ..., m\}$ is normal. We say that an index set $I \subseteq \{1, ..., m\}$ is *negative* if the cone generated by $\{-a^i : i \in I\}$ contains a C-negative vector.

Proposition 2.3 [12, Proposition 3.5]. *A nonempty convex subset $F \subseteq M$ is a face of M if and only if there is a normal subset $I(F) \subseteq \{1, \ldots m\}$ such that F is defined by the system*

$$\begin{aligned} &\langle a^i, x\rangle = b_i, i \in I(F) \\ &\langle a^j, x\rangle \geq b_j, j \in \{1, \ldots, p\}\backslash I(F), \end{aligned} \tag{2}$$

in which case $\dim F = n - \text{rank}\{a^i : i \in I(F)\}$.

Proposition 2.4 [12, Corollary 5.4]. *Let $I(F)$ be the index set determining a face F of M by (2). Then F is an efficient solution face if and only if the set $I(F)$ is negative normal.*

Remark 1. In view of Propositions 2.3 and 2.4, to determine efficient solution faces for (VP) one can search index subsets I of $\{1, 2, ..., m\}$ and verify their normality and negativity.

The following proposition says that verifying negativity of an index subset can be reduced to the existence problem of solutions of a linear inequality system that may be solved by some standard methods.

Proposition 2.5 [12, Proposition 4.2]. *A subset $I \subseteq \{1,\dots,m\}$ is negative if and only if the following system is consistent (has a solution)*

$$\sum_{i\in I}\mu_i a^i = \sum_{j=1}^{p}\lambda_j c^j;\ \ \mu_i \geq 0,\ i\in I;\ \lambda_j > 0,\ j=1,\dots,p. \qquad (3)$$

For a point $x \in M$ we denote by $I(x)$ the set of all active indices at x with respect to the system (1), that is $I(x) := \{i \in \{1,2,\dots,m\} : \langle a^i, x\rangle = b_i\}$. We present here a condition to determine whether E_M is a subset of a face F of the polyhedron M in terms of negative normal index sets.

Proposition 2.6. *Let $I(F)$ be the index set determining a face F of M by* (2). *Assume that F contains at least one efficient solution vertex for* (VP). *Then $E_M \subseteq F$ if and only if for every efficient solution vertex $x \in F$ for* (VP) *the set $I(x) \setminus I(F)$ does not contain any negative normal index subset.*

Proof. *"Only if"*. Suppose that $E_M \subseteq F$. Let $x \in F$ be an efficient solution vertex. If $I(x) \setminus I(F)$ contains a negative normal index set I_0 by Proposition 2.4, I_0 determines an efficient face F_0. Then, since $E_M \subseteq F$, we have $F_0 \subseteq F$. This is impossible, since $I_0 \subset I(x) \setminus I(F)$ and I_0 determines F_0.

"If". Assume that for every efficient solution vertex $x \in F$ for (VP) the index set $I(x) \setminus I(F)$ does not contain any negative normal index subset. Assume on the contrary that there is an efficient point $x^* \notin F$. Since the efficient set E_M is connected, there is a path l composed of some efficient edges connecting x^* to an efficient vertex $\bar{x} \in F$. It means that the path l contains an efficient edge e emanating from the efficient vertex $\bar{x}$ such that $e \not\subset F$. Let I_e be the index set determining the edge e. Clearly, $I_e \subseteq I(\bar{x}) \setminus I(F)$. By Proposition 2.4, I_e is negative normal. This contradicts the assumption. Thus, $E_M \subseteq F$. ∎

3. Cutting Plane and the Reduced Problem

Let x^0 be an efficient solution vertex for (VP). From now on, we denote by $F(x^0)$ the intersection of M and the hyperplane $\langle d, x\rangle = \langle d, x^0\rangle$,

$$F(x^0) := \{x \in M : \langle d, x\rangle = \langle d, x^0\rangle\}.$$

To $F(x^0)$ we associate the polyhedron

$$M(x^0) := \{x \in M : \langle d, x\rangle \leq \langle d, x^0\rangle\}$$

and the reduced problem

$$\text{(RVP)} \qquad \text{Min}\,\{Cx,\ x \in M(x^0)\}.$$

Denote by $E_{M(x^0)}$ the set of all efficient points for (RVP). Note that, in general, neither $E_M \subseteq E_{M(x^0)}$ nor $E_{M(x^0)} \subseteq E_M$. However, on $M(x^0) \setminus F(x^0)$ these two sets coincide. Namely we have the following.

Proposition 3.1. *Let x^0 be an efficient solution vertex for* (VP). *Then*
(i) $E_M \cap (M(x^0) \setminus F(x^0)) = E_{M(x^0)} \cap (M(x^0) \setminus F(x^0))$,
(ii) *If* $x^* \in (M(x^0) \setminus F(x^0)) \cap E_M$, *then there is a path of efficient edges for* (RVP) *connecting* x^* *to* x^0.

Proof. (i) Observe that for every point $x^* \in (M(x^0) \setminus F(x^0))$ the normal cones $N_M(x^*)$ and $N_{M(x^0)}(x^*)$ coincide. The conclusion is immediate from Proposition 2.2.

(ii) Since $x^0 \in E_M$ and $M(x^0) \subseteq M$, one has $x^0 \in E_{M(x^0)}$. On the other hand, by (i) we also have $x^* \in E_{M(x^0)}$. The conclusion is now obtained from the well-known fact [11] that the efficient set of a multiobjective linear programming problem is pathwise connected. ∎

The following theorem, which is an immediate consequence of the above proposition, gives a fundamental relationship between solutions for (P) and the efficient set for the reduced problem (RVP).

Theorem 3.2. *Let x^0 be an efficient solution vertex for* (VP). *Suppose that x^0 is a local optimal solution to* (P). *Then*
(i) x^0 *is a global optimal solution for* (P) *if and only if* $E_{M(x^0)} \subset F(x^0)$,
(ii) *If x^0 is not a global optimal solution for* (P) , *then there is a point* $x^1 \in E_M \cap F(x^0)$ *such that x^1 is not a local optimal solution for* (P) *and there is a path of efficient edges for* (RVP) *lying in* $F(x^0)$ *and connecting* x^0 *to* x^1.

Proof. (i) If x^0 is a global optimal solution for (P), then $(M(x^0) \setminus F(x^0)) \cap E_M = \emptyset$. Therefore, by Proposition 3.1(i), $(M(x^0) \setminus F(x^0)) \cap E_{M(x^0)} = \emptyset$. This means that $E_{M(x^0)} \subseteq F(x^0)$.

Now, assume that $E_{M(x^0)} \subseteq F(x^0)$. Then $(M(x^0) \setminus F(x^0)) \cap E_{M(x^0)} = \emptyset$. By Proposition 3.1(i), $(M(x^0) \setminus F(x^0)) \cap E_M = \emptyset$. It means that there is no an efficient solution x^* for (VP) such that $\langle d, x^* \rangle < \langle d, x^0 \rangle$. Therefore, x^0 is a global optimal solution for (P).

(ii) Since x^0 is not an optimal solution for (P), there is a point $x^* \in (M(x^0) \setminus F(x^0)) \cap E_M$. Therefore, by Proposition 3.1(ii) there is a path $L \subset E_{M(x^0)}$, which is composed of edges $[x^0, y^1], [y^1, y^2], \ldots, [y^k, x^*]$. Because x^0 is a local solution for (P), $[x^0, y^1] \subset F(x^0)$. Let i be the first index such that $y^{i+1} \in M(x^0) \setminus F(x^0)$. By Proposition 3.1 (i) and the closedness of the efficient set for (VP) it follows that $[y^i, y^{i+1}] \subset E_M$. Set $x^1 := y^i$. Then x^1 is the desired point. ∎

The following corollary is immediate from Theorem 3.2.

Corollary 3.3. *If there is no efficient edge for* (*RVP*) *lying on* $F(x^0)$ *and emanating from* x^0 (i.e. $E_{M(x^0)} = \{x^0\}$), *then x^0 is a global optimal solution for* (P).

From now on, for a fixed local solution vertex x^0 of (P) we will denote $a^0 := -d$ and $b_0 := -\langle d, x^0 \rangle$. Then $M(x^0)$ is the solution set of the system

$$\langle a^i, x \rangle \geq b_i, i = 0, 1, \ldots, m. \tag{4}$$

In general, there may be some redundant inequalities in this system. In this case we can move out all of them. Namely, let

$$\alpha_i := \min\{\langle d, x\rangle : x \in M, \quad \langle a^i, x\rangle = b_i\}, \; i = 1, 2, \dots, m$$

and

$$I_{M(x^0)} := \{i \in \{1, \dots, m\} : \alpha_i < \langle d, x^0\rangle\} \cup \{0\}.$$

Proposition 3.4. *For a given local optimal solution vertex x^0 for* (P) *the polyhedron $M(x^0)$ is the solution set of the system*

$$\langle a^i, x\rangle \geq b_i, i \in I_{M(x^0)}. \tag{5}$$

In this system there is no redundant inequality, except the trivial case when x^0 is an extremal point of the function d over M.

Proof. Since there are no redundant equalities in the system (1) which determine the polyhedron M, every set $\{x \in M, \langle a^i, x\rangle = b_i\}$, $i = 1, 2, \dots, m$ is a facet of M. Then the conclusion is immediate from the definition of $M(x^0)$. ■

In view of Proposition 3.4, in the sequel we may always assume that $M(x^0)$ is determined by (5). For a point $x \in M(x^0)$ we denote by $II(x)$ the set of all active indices at x with respect to the system (5), i.e.,

$$II(x) := \{i \in I_{M(x^0)} : \langle a^i, x\rangle = b^i\}.$$

4. The Algorithm for Solving (P)

4.1. Algorithm

The algorithm for solving problem (P) can be outlined as follows. Its implementation will be described in detail in the next part.

Initialization Step. Check whether $E_M = \emptyset$.

(a) If Yes, the problem (P) has no feasible solution. The algorithm is terminated.

(b) Else, find an optimal solution x' of the problem

$$\min\{\langle d, x\rangle : x \in M\}.$$

(b1) If $x' \in M_E$ then x' is a global solution of (P), the algorirthm is terminated.

(b2) Otherwise, find an initial efficient vertex x^0 for (VP). Go to Step 1.

Step 1. (Finding a Local Optimal Solution of (P))
Starting with x^0, determine an adjacent efficient edge for (VP) yielding a strict decrease in $\langle d, x\rangle$.

(a) If there is no such efficient edge, x^0 is a local optimal solution of (P). Go to Step 2.

(b) Else,

(b1) If the efficient edge is a ray, the linear function $\langle d, .\rangle$ tends to $-\infty$ on this edge (Problem (P) has no finite optimal value.) The algorithm is terminated.
(b2) Otherwise, say x^* is the other end of this efficient edge, set $x^0 \leftarrow x^*$ and return to Step 1.

Step 2. (Finding a better efficient vertex or show that the current efficient vertex x^0 is a global optimal solution of (P))
Search on $F(x^0)$ to find an efficient solution vertex $\bar{x}$ for (RVP) which has an adjacent efficient edge L yielding a strict decrease in $\langle d, x\rangle$.
(a) If no such point $\bar{x}$ exists, the vertex x^0 is a global optimal solution for (P) (Theorem 3.2). The algorithm is terminated.
(b) Else,
(b1) If the efficient edge L is a ray, the linear function $\langle d, .\rangle$ tends to $-\infty$ on this edge (Problem (P) has no finite optimal value.) The algorithm is terminated.
(b2) Otherwise, say x^* is the other end of this efficient edge L, set $x^0 \leftarrow x^*$ and return Step 1.

4.2. Implementation of the Algorithm

In initialization Step, to check whether $E_M \neq \emptyset$ one can carry out the procedure proposed in [2, 12]. In the case when the efficient set is not empty, several methods can be used in order to find the first efficient vertex x^0 (see, for example, in [2, 7, 12]).

4.2.1. Step 1. Let x^0 be a given efficient vertex for (VP). The following proposition shows that the determining a local optimal solution of (P) can be carried out by considering index subsets of $II(x^0)$.

Proposition 4.2. *Let x^0 be an efficient solution vertex for* (VP). *Then x^0 is not a local optimal solution for* (P) *if and only if there exists an index subset $I_0 \subset II(x^0) \setminus \{0\}$ with $|I_0| = n - 1$ such that the vectors $\{a^i : i \in I_0\}$ are linear independent and I_0 is negative normal set with respect to* (RVP). *For such an index set I_0 the set $\{x \in M(x^0) : \langle a^i, x\rangle = b_i,\ i \in I_0\}$ is an efficient edge for* (VP), *emanating from x^0 and yielding a strict decrease in $\langle d, x\rangle$. In particular, if $II(x^0) \setminus \{0\}$ is not negative, x^0 must be a local optimal solution for* (P).

Proof. Observe that x^0 is not a local optimal solution for (P) if and only if there exists an efficient edge $L \subset E_M$ emanating from x^0 and yielding a decrease in $\langle d, x\rangle$. By Proposition 3.1(i) such an edge L must be an edge of $E_{M(x^0)}$. In view of Proposition 2.4 and definitions, this is equivalent to that there exists an index subset $I_0 \subset II(x^0)$ with $|I_0| = n - 1$ such that the vectors $\{a^i : i \in I_0\}$ are linear independent, I_0 is negative normal with respect to (RVP) and $L = \{x \in M(x^0) : \langle a^i, x\rangle = b_i, i \in I_0\}$. Since L yields a decrease in $\langle d, x\rangle$, $L \not\subset F(x^0)$, and hence, $0 \notin I_0$. ∎

This step is implemented by the following Procedure $LS(x^0)$.

Procedure LS(x^0);

Input - an efficient vertex x^0 of E_M.

Output Either conclude $\min\{\langle d,x\rangle : x \in E_M\} = -\infty$ or give a local optimal solution vertex of (P).

Let $0 \leftarrow j$.

Iteration j $(j = 0,1,2\dots)$.

Check whether the index set $II(x^0) \setminus \{0\}$ is negative .

(a) If not, then x^0 is a local solution for (P) (Proposition 4.2). The procedure is terminated. Go to Step 2 of the algorithm.

(b) Else, find a subset $I_0 \subseteq II(x^0)\setminus\{0\}$ with $|I_0| = n-1$ and $\{a^i, i \in I_0\}$ linearly independent such that I_0 is negative normal with respect to (RVP).

(b1) If such index sets do not exist, x^0 is a local solution for (P) (Proposition 4.2). The procedure is terminated. Go to Step 2 of the algorithm.

(b2) Otherwise, determine the edge $L := \{x \in M(x^0) : \langle a^i, x\rangle = b_i, i \in I_0\}$.

- If L is a ray, the linear function $\langle d, .\rangle$ tends to $-\infty$ on this edge (Problem (P) has no finite optimal value). Terminate the algorithm.
- Otherwise, $L = [x^0, x^*]$. Let $x^0 \leftarrow x^*$. Set $j \leftarrow j+1$ and go to iteration j.

4.2.2. Step 2. This is the most complicated step of the algorithm. In this step, starting from a current local optimal solution x^0 for (P) we have to find a better efficient vertex or show that the current efficient vertex x^0 is a global optimal solution of (P). To do it we will search along paths emanating from x^0 and lying in $E_{M(x^0)} \cap F(x^0)$ to find an efficient vertex $\bar{x}$ for (RVP) that has an adjacent efficient edge yielding a decrease in $\langle d,x\rangle$. In view of Theorem 3.2 the local optimal solution x^0 is not a global optimal solution for (P) if such point $\bar{x}$ exists.

Let x^1 be an efficient vertex for (RVP) that lies in a considered path and is a local minimal point of the function $\langle d,x\rangle$ over $E_{M(x^0)}$. We will search on the face $F(x^0)$ an efficient vertex $\bar{x}$ for (RVP) incident to x^1 and check whether $\bar{x}$ is a local minimal point of the function $\langle d,x\rangle$ over $E_{M(x^0)}$. Note from definitions that $\bar{x}$ is not a local minimal point of the function $\langle d,x\rangle$ over $E_{M(x^0)}$ if and only if there exists an efficient edge L for (RVP) emanating from $\bar{x}$ and lying in $M(x^0) \setminus F(x^0)$. Then, by an argument analogous to that used in the proof of Proposition 4.2, we have

Proposition 4.3. *Let x^0 be a local optimal solution for* (P) *and $\bar{x} \in F(x^0)$ an efficicent vertex for* (RVP). *Then $\bar{x}$ is not a local minimal point of the function $\langle d,x\rangle$ over $E_{M(x^0)}$ if and only if there exists an index subset $I_0 \subset II(\bar{x}) \setminus \{0\}$ with $|I_0| = n-1$ such that the vectors $\{a^i : i \in I_0\}$ are linear independent and I_0 is negative and normal with respect to* (RVP). *For such an index set I_0 the set $L := \{x \in M(x^0) : \langle a^i, x\rangle = b_i, i \in I_0\}$ is an efficient linear path for* (VP) *emanating from $\bar{x}$ and yielding a strict decrease in $\langle d,x\rangle$.*

This step is implemented by the following Procedure CP(x^0).

Procedure $CP(x^0)$;

Input - a local optimal solution vertex x^0 for (P);
Output - Either conclude $\min\{\langle d, x\rangle : x \in E_M\} = -\infty$, conclude x^0 is a global optimal solution for (P) or give a better efficient vertex for (VP) lying in $M(x^0) \setminus F(x^0)$.

Set $V_0 := \emptyset$, $V_1 := \{x^0\}$ and $j \leftarrow 0$.
Iteration j $(j = 0, 1, 2, \dots)$.

- If $V_1 = \emptyset$, then x^0 is a global optimal solution of (P) and the algorithm is terminated.
- Otherwise, take a point $x^1 \in V_1$. Set $k_j \leftarrow 0$.

Sub-Iteration k_j $(k_j = 0, 1, 2 \dots)$.
Find an efficient vertex $\bar{x}$ incident to x^1 such that $\bar{x} \in F(x^0)$ and $\bar{x} \notin V_0 \cup V_1$.

(a) If such vertices do not exist, let $V_1 := V_1 \setminus \{x^1\}$, $V_0 := V_0 \cup \{x^1\}$ and terminate Sub-Iteration k_j. Let $j := j + 1$ and go to Iteration j.

(b) Otherwise,

(b1) If $II(\bar{x}) \setminus \{0\}$ is not negative, $\bar{x}$ is a local minimal point solution of the function $\langle d, x\rangle$ over $E_{M(x^0)}$. Add $\bar{x}$ to V_1. Let $k_j \leftarrow k_j + 1$ and go to Sub-Iteration k_j.

(b2) Otherwise, search a subset $I_0 \subseteq II(\bar{x}) \setminus \{0\}$ with $|I_0| = n - 1$ and $\{a^i, i \in I_0\}$ linearly independent such that I_0 is negative normal with respect to (RVP).

(b2.1) If such index sets do not exist, $\bar{x}$ is a local minimal point solution of the function $\langle d, x\rangle$ over $E_{M(x^0)}$. Add $\bar{x}$ to V_1. Let $k_j \leftarrow k_j + 1$ and go to Sub-Iteration k_j.

(b2.2) Otherwise, the point $\bar{x}$ is not a local minimal point solution of the function $\langle d, x\rangle$ over $E_{M(x^0)}$. Then, determine the efficient edge

$$L := \{x \in M(x^0) : \langle a^i, x\rangle = b_i, i \in I_0\}.$$

$*$ If L is a ray, the linear function $\langle d, .\rangle$ tends to $-\infty$ on this edge (Problem (P) has no finite optimal value). Terminate the algorithm.

$*$ Otherwise, $L = [x^0, x^*]$. Let $x^0 \leftarrow x^*$. Terminate the procedure and go to Step 1 of the algorithm.

4.3. Some Remarks on the Algorithm

We conclude this section with the following comments.

Proposition 4.4. *This algorithm solves problem* (P) *in a finite number of steps.*

Proof. Indeed, the number of improving steps on efficient edges is finite, since in the calculation process the objective function decreases, any efficient edge of M can occur at most only once. As the polyhedron M has a finite number of edges, the algorithm proposed for solving (P) is finite. ■

Remark 2. In the above algorithm, we often solve the problem of finding an efficient edge and vertex adjacent to a given efficient vertex x^0. This can be done by searching and finding index subsets I_0 of the active index set of x^0 such that I_0 is negative normal, $|I_0| = n-1$ and $\{a^i, i \in I_0\}$ are linearly independent (Proposition 2.4). Each of such index subsets I_0 will determine an efficient edge incident to x^0, and gives an adjacent vertex if this edge is bounded. This problem can be solved by a quite simple procedure presented in [12].

Remark 3. In the case of finding an efficient vertex $\bar{x} \in F(x^0)$ that emanates from a given efficient vertex $x^1 \in F(x^0)$, we need only to search and find index subsets $I_1 \subset II(x^1)$ containing the zero index such that $|I_1| = n-1$, $\{a^i, i \in I_1\}$ are linearly independent and I_1 is negative normal.

Remark 4. The above algorithm is a process of finite steps. In each of steps we work with an efficient vertex x^0 and the associated polyhedron $M(x^0)$ determined by the system (5). It is worth to note that the number $|I_{M(x^0)}|$ of inequalities in the system (5), which depends on x^0 as in Proposition 3.4, decreases after each steps.

5. Examples

The following examples have been computed by a program written in DELPHI 2.0.

Example 1. Consider the problem $\min\langle d, x\rangle$, s.t. $x \in E_M$. Here, $d = (1, 0)$ and E_M is the efficient set of the following linear multiobjective programming problem

$$\text{Min}\begin{bmatrix} -x_1 + 3x_2 \\ -x_1 - 3x_2 \end{bmatrix}, \text{ s.t. } x \in M,$$

$$M = \{x \in R^2 \mid -x_1 - 2x_2 \geq -8, -2x_1 - x_2 \geq -7, -x_1 + 2x_2 \geq -1, x_1, x_2 \geq 0\}.$$

In this example the calculation process is descibed as follows.

Initialization. By the procedure proposed in [12], we obtain an initial efficient extreme solution $x^0 = (3, 1)$ for (VP).

Iteration 1. Starting from $x^1 = (3, 1)$, using the procedure given in [12], we obtain the efficient extreme solution $x^2 = (1, 0)$ which is a local optimal solution for (P). Then, searching on $F(x^2) := \{x \in M : \langle d, x\rangle = 1\}$, we obtain the efficient vertex $x^3 = (1, 3.5) \in F(x^2)$ for (RVP) and the efficient vertex $x^4 = (0, 4) \in (M(x^0) \setminus F(x^0))$ incident to x^3.

Iteration 2. Start from $x^4 = (0, 4)$. There is no efficient edge emanating from x^4 yielding a decrease in the objective function. Considering $F(x^4) := \{x \in M : \langle d, x\rangle = 0\}$. There is no efficient point in $F(x^4)$ that has an adjacent efficient edge yieding a decrease in $\langle d, x\rangle$. So, x^4 is a global optimal solution for (P).

For convenience, we will illustrate the calculation process by the directed graph:

$$x^1 = (3, 1) \to x^2 = (1, 0) \to x^3 = (1, 3.5) \to |x^4 = (0, 4).$$

Here, x^1 is the starting point and the notation $|x$ indicates a global optimal solution for (P).

Example 2. Consider the linear multiobjective programming problem

$$\text{Min} \begin{bmatrix} -x_1 + 0x_2 + 0x_3 \\ 0x_1 - x_2 + 0x_3 \end{bmatrix}, \quad \text{s.t.} \quad x \in M$$

$M = \{x \in R^3 \mid 2x_1 + x_2 \le 16, 8x_1 + 5x_2 \le 66, 2x_1 + 3x_2 \le 27, x_1 \ge 0, 0 \le x_2 \le 7, 0 \le x_3 \le 2\}$.

The problem (P) that we want to solve is $\min(x_1 + x_2)$, s.t. $x \in E_M$.

We obtain the following graph

$$x^1 = (4.5, 6, 2) \to x^2 = (3, 7, 2) \to$$
$$\to x^3 = (5.333, 4.667, 0) \to x^4 = (7, 2, 0) \to x^5 = (8, 0, 0).$$

Example 3. Consider the linear multiobjective programming problem

$$\text{Min} \begin{bmatrix} -1 & -3 & 2 & 0 & -1 \\ -3 & 1 & 0 & -3 & -1 \\ -1 & 0 & -2 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_5 \end{bmatrix},$$

$$\text{s.t.} \quad \begin{bmatrix} -2 & -4 & 0 & 0 & -3 \\ 0 & 0 & -2 & -5 & -4 \\ -5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ -5 & -5 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_5 \end{bmatrix} \ge \begin{bmatrix} -27 \\ -35 \\ -26 \\ -24 \\ -36 \end{bmatrix},$$

$$x_1, \dots, x_5 \ge 0.$$

For the problem $\min -x_3$, s.t. $x \in E_M$, we obtain the graph

$$x^1 = (5.2, 0, 0, 2.573, 5.533) \to x^2 = (5.2, 0, 5, 0.573, 5.533) \to$$
$$\to x^3 = (4.826, 0, 5.935, 0, 5.783) \to |x^4 = (0.2, 0, 17.5, 0, 0).$$

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