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Is a Right Perfect Right Self-Injective Ring Right PF?*

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Abstract. Several equivalent formulations of the question of the title are given. In particular, it is proved that a right perfect right self-injective ring R is right PF if and only if $R/Z(_RR)$ is a semiprime ring. It is shown that a right perfect left quasi-continuous ring is left and right Kasch. Some necessary and sufficient conditions for a semiperfect ring to be right Kasch are given.

Throughout this paper all rings are associative with identity and all modules are unital. If R is a ring, the Jacobson radical of R is denoted by $J(R)$, the singular ideals are denoted by $Z(_RR)$ and $Z(R_R)$ and socles are denoted by $\mathrm{Soc}(_RR)$ and $\mathrm{Soc}(R_R)$. The left (respectively right) annihilator of a subset X of R is denoted by $l_R(X)$ (resp. $r_R(X)$). We write $N \trianglelefteq M$ to mean that N is an essential submodule of M. The socle of a module M is denoted by $\mathrm{Soc}(M)$.

A ring R is called *right pseudo-Frobenius* (PF) if R_R is an injective cogenerator and *right Kasch* if R_R contains a copy of each simple right R-module. A right self-injective right Kasch ring is right PF ([1, Proposition 18.15]). A ring R is said to be *left principally injective* (respectively *left mininjective*) if every R-homomorphism from a principal (resp. minimal) left ideal of R can be extended to R. A ring R is left principally injective if and only if every principal right ideal is a right annihilator ([16, Lemma 1.1]). In a right cogenerator ring R every right ideal is a right annihilator ([11, Lemma 12.4.1]) and so R is left principally injective.

From Osofsky [18] and Kato [12] it follows that a one sided perfect and two sided self-injective ring is quasi-Frobenius (QF) (i.e., left and right PF and left

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and right Artinian). It is an open problem whether a semiprimary right self-injective ring is QF (see [6, Exercise 24.16, p. 218]). D. V. Huynh [10] "proved" that a right perfect right self-injective ring is QF but later he discovered a gap in the proof (see [4, Remarks, p. 539]). Many authors have studied this question and have given affirmative answers under some extra conditions (see e.g., [2, 5, 9 21]).

It is well known that a left perfect right self-injective ring is right PF ([7, Corollary 4.21]). But whether a right perfect right self-injective ring is right PF, is an open problem (see [21, Remark 2, p. 754]). Osofsky [19] gave an example of a semiperfect right self-injective ring which is not right PF.

The following result is well known (see e.g. [11, Lemma 12.4.1] and [16, Theorem 2.3 and Corollary 2.2]).

Proposition 0.1. *Let R be a right PF ring. Then*
(a) *R is left principally injective,*
(b) $J(R) = Z(_RR)$,
(c) $\mathrm{Soc}(_RR) = \mathrm{Soc}(R_R)$,
(d) $\mathrm{Soc}(R_R) \trianglelefteq {_R}R$.
(e) *Every simple left R-module embeds in* $\mathrm{Soc}(R_R)$.

This makes one ask the following questions for a right perfect right self-injective ring R:
(*i*) *Is R left mininjective?*
(*ii*) *Is R left principally injective?*
(*iii*) *Is $J(R) = Z(_RR)$?*
(*iv*) *Is* $\mathrm{Soc}(_RR) = \mathrm{Soc}(R_R)$?
(*v*) *Is* $\mathrm{Soc}(R_R) \trianglelefteq {_R}R$?
(*vi*) *Does every simple left R-module embed in* $\mathrm{Soc}(R_R)$?

In Sec. 2, we prove that all the questions listed above (i.e., (i) to (vi)) are equivalent to the question: *Is a right perfect right self-injective ring right PF?* We also prove that a right perfect right self-injective ring R is right PF if and only if $R/Z(_RR)$ is a semiprime ring. In Sec. 3, it is shown that a right perfect left quasi-continuous ring is left and right Kasch. As a corollary it follows that a left quasi-continuous right perfect right self-injective ring is right PF.

In Sec. 1, we study semiperfect Kasch rings.

1. Semiperfect Kasch Rings

Lemma 1.1 [1, Proposition 15.17]. *For a ring R with radical J the following are equivalent:*
(a) *R is semilocal;*
(b) *for every right R-module M,* $\mathrm{Soc}(M) = l_M(J)$.

Lemma 1.2. *Let e be an idempotent of a semilocal ring R with $J = J(R)$. Then,*

$$\mathrm{Hom}_R(eR/eJ, R) \cong l_R(J)e = \mathrm{Soc}(R_R)e.$$

Proof. It is easily checked that $\lambda : Re \to \mathrm{Hom}_R(eR, R)$ defined as $\lambda(re)(es) = res$, where $r, s \in R$, is an R-isomorphism (see [1, Proposition 4.6]). Thus by Lemma 1.2,

$$\mathrm{Hom}_R(eR/eJ, R) \cong l_{Re}(J) = l_R(J) \cap Re = l_R(J)e = \mathrm{Soc}(R_R)e.$$

This completes the proof. ∎

Equivalence of assertions (a) and (b) in the following proposition improves upon [17, Proposition 2.3 (2)].

Proposition 1.3. *For a semiperfect ring R with $J = J(R)$ the following statements are equivalent:*
(a) *R is right Kasch;*
(b) $\mathrm{Soc}(R_R)e \neq 0$ *for every primitive idempotent e of R;*
(c) $r_R l_R(J) = J$;
(d) $r_R(\mathrm{Soc}(R_R)) = J$.
In particular, these equivalent conditions imply that for every nonzero idempotent e of R, $Hom_R(eR/eJ, R) \neq 0$.

Proof. Equivalence of (a) and (b) follows from Lemma 1.2 and equivalence of (c) and (d) follows from Lemma 1.1.
(a) ⇒ (d). As R is right Kasch, $\mathrm{Soc}(R_R)$ contains a copy of each simple right R-module. Also as the Jacobson radical of a ring R coincides with the intersection of annihilators in R of all simple right R-modules, (d) follows.
(d) ⇒ (b). For any idempotent e of R, $\mathrm{Soc}(R_R)e = 0$ implies $e \in r_R(\mathrm{Soc}(R_R)) = J$. As the Jacobson radical of a ring does not contain any non-zero idempotent so $e = 0$. Thus (b) follows. This completes the proof. ∎

The following result, which now follows as a corollary, was proved by Nicholson and Yousif in [15, Lemma 3].

Corollary 1.4. *A semiperfect ring R with* $\mathrm{Soc}(R_R) \trianglelefteq_R R$ *is right Kasch.*

Proof. For every primitive idempotent e of R, $\mathrm{Soc}(R_R)e = \mathrm{Soc}(R_R) \cap Re \neq 0$. Hence the result follows from Proposition 1.3. ∎

If $L \leq K$ are submodules of a module M with L maximal in K, then K/L is called a *composition factor* of M.

Lemma 1.5 [1, Exercise 27.9]. *Let R be a ring and e be an idempotent of R with eR/eJ a simple right R-module, where $J = J(R)$. Then a right R-module M has a composition factor isomorphic to eR/eJ if and only if $Me \neq 0$.*

Proposition 1.6. *Let R be a semiperfect ring with* $\mathrm{Soc}(_RR) \subseteq \mathrm{Soc}(R_R)$. *Then the following conditions are equivalent:*

(a) $\mathrm{Soc}(Re) \neq 0$ *for every primitive idempotent* e *of* R;
(b) *Every simple right* R*-module embeds in* $\mathrm{Soc}({}_RR)$.
In particular, these conditions imply that R *is right Kasch.*

Proof. (a) $\Rightarrow$ (b). As $\mathrm{Soc}({}_RR)e = \mathrm{Soc}({}_RR) \cap Re = \mathrm{Soc}(Re) \neq 0$, in view of Lemma 1.5, $\mathrm{Soc}({}_RR)$ contains a composition factor isomorphic to eR/eJ for every primitive idempotent e of R. As $\mathrm{Soc}({}_RR) \subseteq \mathrm{Soc}(R_R)$, $\mathrm{Soc}({}_RR)$ is a semisimple right ideal of R. Hence there exists an embedding $eR/eJ \to \mathrm{Soc}({}_RR)$ for every primitive idempotent e of R and thus (b) follows.
(b) $\Rightarrow$ (a). By Lemma 1.5, $0 \neq \mathrm{Soc}({}_RR)e = \mathrm{Soc}(Re)$ for every primitive idempotent e of R. This completes the proof. ∎

A module M is said to satisfy:
the (C_1)-condition if every submodule of M is essential in a direct summand of M;
the (C_3)-condition if whenever M_1 and M_2 are direct summands of M with $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is also a direct summand of M.

A module satisfying condition (C_1) is called *CS*; if it satisfies (C_1) and (C_3) it is called *quasi-continuous*. A quasi-injective module is quasi-continuous (see [13, Proposition 2.2]).

Lemma 1.7 [13, Proposition 2.5]. *An indecomposable module is CS if and only if it is uniform.*

Nicholson and Yousif [16, Theorem 2.4] proved that a semiperfect right principally injective ring is right Kasch if and only if $\mathrm{Soc}(R_R) \trianglelefteq {}_RR$. We prove

Proposition 1.8. *Let* R *be a semiperfect ring with* ${}_RR$ *CS. Then* R *is right Kasch if and only if* $\mathrm{Soc}(R_R) \trianglelefteq {}_RR$.

Proof. If $\mathrm{Soc}(R_R) \trianglelefteq {}_RR$ then R is right Kasch by Corollary 1.4. Now suppose that R is right Kasch. Let $e_1, ..., e_n$ be a complete orthogonal set of primitive idempotents of R. Then $R = \oplus_{i=1}^n Re_i$ and each Re_i is uniform by Lemma 1.7. By Proposition 1.3(b), $\mathrm{Soc}(R_R)e_i = \mathrm{Soc}(R_R) \cap Re_i \neq 0$ for each i. Then $\mathrm{Soc}(R_R) \cap Re_i \trianglelefteq Re_i$ for each i and so $\oplus_{i=1}^n (\mathrm{Soc}(R_R) \cap Re_i) \trianglelefteq \oplus_{i=1}^n Re_i = {}_RR$. Since $\oplus_{i=1}^n (\mathrm{Soc}(R_R) \cap Re_i) \subseteq \mathrm{Soc}(R_R)$ we get $\mathrm{Soc}(R_R) \trianglelefteq {}_RR$. This completes the proof. ∎

2. Right Perfect Right Self-Injective Rings

We first list the following characterizations of a right PF ring. These are due to Azumaya [3], Gomez Pardo and Guil Asensio [8], Osofsky [18] and Utumi [20].

Theorem 2.1. *The following statements are equivalent for a ring* R:
(i) R *is right PF*;
(ii) R_R *is a CS cogenerator*;

(iii) *R is semiperfect right self-injective with* $\text{Soc}(R_R) \trianglelefteq R_R$;
(iv) *Every faithful right R-module is a generator*;
(v) R_R *is injective and* $\text{Soc}(R_R)$ *is a finitely generated essential right ideal.*

Lemma 2.2 *Let R be a semilocal ring with* $J(R) \subseteq Z({}_RR)$. *Then* $\text{Soc}({}_RR) \subseteq \text{Soc}(R_R)$.

Proof. As the socle of a module is the intersection of all its essential submodules, for any $r \in Z({}_RR)$, $\text{Soc}({}_RR) \subseteq l_R(r)$ because by definition $l_R(r) \trianglelefteq {}_RR$. Thus using Lemma 1.1, we get

$$\text{Soc}({}_RR) \subseteq \bigcap_{r \in Z({}_RR)} l_R(r) = l_R(Z({}_RR)) \subseteq l_R(J(R)) = \text{Soc}(R_R).$$

Lemma 2.3 [17, Theorem 1.14(4)]. *For a left mininjective ring R*, $\text{Soc}({}_RR) \subseteq \text{Soc}(R_R)$.

Theorem 2.4. *Let R be a semiperfect right self-injective ring with* $\text{Soc}(Re) \neq 0$, *for every primitive idempotent e of R. Then the following conditions are equivalent:*
(a) *R is right PF*;
(b) *R is left principally injective*;
(c) *R is left mininjective*;
(d) $J(R) = Z({}_RR)$;
(e) $\text{Soc}({}_RR) = \text{Soc}(R_R)$;
(f) $\text{Soc}(R_R) \trianglelefteq {}_RR$;
(g) *Every simple left R-module embeds in* $\text{Soc}(R_R)$.

Proof.

(a) ⇒ (b), (c), (d), (e), (f) and (g). Well known (see e.g., [16, Corollary 2.2 and Theorem 2.3]).

(b) ⇒ (e) and (c)⇒ (e). Follow from Lemma 2.3.

(d) ⇒ (e). Follows from Lemma 2.3 and Lemma 2.2.

(f) ⇒ (e). As $\text{Soc}(R_R) \trianglelefteq {}_RR$ so $\text{Soc}({}_RR) \subseteq \text{Soc}(R_R)$. Also by Lemma 2.3, $\text{Soc}({}_RR) \supseteq \text{Soc}(R_R)$. This gives (e).

(e) ⇒ (a). Follows from Proposition 1.6.

(g) ⇒ (a). Let $e_1, \ldots, e_n$ be a complete orthogonal set of primitive idempotents of R. As R is right self-injective, $\text{Soc}(R_R) \subseteq \text{Soc}({}_RR)$ by Lemma 2.3. Since every simple left R-module embeds in $\text{Soc}(R_R)$, by Proposition 1.6 we get $\text{Soc}(e_iR) \neq 0$ for every $1 \leq i \leq n$. Now as e_iR is uniform (see Lemma 1.7), $\text{Soc}(e_iR) \trianglelefteq e_iR$ for every $1 \leq i \leq n$. But then $\text{Soc}(R_R) = \oplus_{i=1}^{n} \text{Soc}(e_iR) \trianglelefteq \oplus_{i=1}^{n} e_iR = R_R$. Thus R is right PF (see Theorem 2.1(iii)). This completes the proof. ∎

The proof of the following lemma is easy and hence is omitted.

Lemma 2.5. *If X is a right T-nilpotent subset of a ring R then for any nonzero left R-module M*, $r_M(X) \neq 0$.

A ring is said to be *semiprime* if it has no nonzero one sided nilpotent ideals.

Proposition 2.6. *A right perfect right self-injective ring R is right PF if and only if $R/Z(_RR)$ is a semiprime ring.*

Proof. Let $J = J(R)$. If R is right PF then $J = Z(_RR)$ and so $R/Z(_RR)$ is semiprime.

Conversely, suppose that $R/Z(_RR)$ is semiprime. In view of Theorem 2.4, it is enough to show that $J = Z(_RR)$. As R is right self-injective, $Z(_RR) \subseteq J$ (see [20, Theorem 1.3]). Suppose, on the contrary, $Z(_RR) \neq J$. As J is right T-nilpotent so $J/Z(_RR)$ is a nonzero right T-nilpotent ideal of $R/Z(_RR)$. By Lemma 2.5, $T = r_{J/Z(_RR)}(J/Z(_RR))$ is a nonzero ideal of $R/Z(_RR)$. As $T^2 = 0$ this violates the semiprimeness of $R/Z(_RR)$. This completes the proof. ∎

3. Quasi-Continuous Kasch Rings

Lemma 3.1 [14, Theorem 4]. *Let M be a quasi-continuous module and let $A_i \trianglelefteq P_i \leq M$, (i=1,2), where P_1 and P_2 are direct summands. Then*

$$A_1 \cong A_2 \Rightarrow P_1 \cong P_2.$$

Corollary 3.2. *A semiperfect left quasi-continuous ring R with* $\mathrm{Soc}(_RR) \trianglelefteq {}_RR$ *is left Kasch.*

Proof. Let $e_1, ..., e_n$ be a basic set of idempotents of R. From Lemma 1.7 it follows that each Re_i is uniform. As $\mathrm{Soc}(Re_i) \neq 0$, $\mathrm{Soc}(Re_i)$ is simple for each $1 \leq i \leq n$. Also in view of Lemma 3.1, $\mathrm{Soc}(Re_i) \not\cong \mathrm{Soc}(Re_j)$ if $i \neq j$. Thus $\{\mathrm{Soc}(Re_1), ..., \mathrm{Soc}(Re_n)\}$ is a complete irredundant set of representatives of simple left R-modules. Hence R is left Kasch and the proof is complete. ∎

Lemma 3.3 (Clark and Huynh [4, Lemma 6(ii)]). *Let R be a quasi-continuous left or right perfect ring. Then $J(R) = Z(_RR)$.*

Proposition 3.4. *A right perfect left quasi-continuous ring R is left and right Kasch.*

Proof. By Corollary 3.2, R is left Kasch. As $J(R) = Z(_RR)$ (see Lemma 3.3), $\mathrm{Soc}(_RR) \subseteq \mathrm{Soc}(R_R)$ (see Lemma 2.2). Hence $\mathrm{Soc}(R_R) \trianglelefteq {}_RR$ and so R is right Kasch by Corollary 1.4. ∎

Corollary 3.5. *A left quasi-continuous right perfect right self injective ring is right PF.*

As a right PF ring is left Kasch the following question arises:

Question 3.6. *Is a right perfect right self-injective ring left Kasch?*

In Theorem 2.4, we proved that if R is a right perfect right self-injective ring such that every simple left R-module embeds in $\mathrm{Soc}(R_R)$ then R is right PF. So one may ask the following stronger question.

Question 3.7. *Is a right perfect right self-injective left Kasch ring right PF?*

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