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On the L-Minimizing Networks in $\mathbb{R}^n$

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Abstract. In this paper we use calibration method to study an absolute minimality of networks in $\mathbb{R}^n$ with respect to the lagrangian of degree 1. The main results of the paper can be used to find the networks minimizing a given parallel convex lagrangian in $\mathbb{R}^n$.

1. Introduction

In the last few years, the networks of least length in $\mathbb{R}^n$ with fixed ends M have been studied by many authors, see for example [3-8]. In [3] the authors used the calibration method to study the global minimality for Steiner networks with respect to the Euclidean metric in $\mathbb{R}^n$. In this paper, by using calibration system we prove that every locally L-minimal network is also absolutely L-minimal in the class of networks of a fixed topological type. We also find conditions for a network to be absolutely L-minimal in the class of all the networks with fixed boundary points. The calibration systems were used first in [5].

2. Locally L-Minimal Networks are Also Absolutely L-Minimal in the Class of Networks with the Same Topological Type in $\mathbb{R}^n$

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with scalar product $(\cdot)$ and norm $|.|$. The tangent space $\mathbb{R}^n_x$ to $\mathbb{R}^n$ at x can be identified with $\mathbb{R}^n$. We denote the vector space of all real differentiable 1-forms on $\mathbb{R}^n$ by $\Omega^1\mathbb{R}^n$.

Suppose that γ is a differentiable 1-dimensional curve $\gamma:\ [a,b]\to\mathbb{R}^n,\ s\to\gamma(s),\ |\vec{\gamma}|=1$. By definition, for every γ,

$$\gamma(w) = \int_\gamma w(s, \vec{\gamma}'(s))ds, \quad w \in \Omega^1\mathbf{R}^n,\ s \in \gamma.$$

Definition 1. *A lagrangian of degree 1 on* $\mathbb{R}^n$ *is a continuous mapping*

$$l : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (x, \xi) \to l(x, \xi)$$

satisfying

1. $l(x, a\xi) = |a| l(x, \xi); \quad \forall x, \xi \in \mathbb{R}^n,\ a \in \mathbb{R},$
2. $l(x, \xi) > 0; \quad \forall \xi \neq 0,\ \forall x \in \mathbb{R}^n.$

Every lagrangian l of degree 1 on $\mathbb{R}^n$ defines a positively homogeneous functional on the set of all the curves

$$L(\gamma) = \int_\gamma l(s, \vec{\gamma}'(s))ds.$$

Obviously, if l is the norm functional $|\cdot|$, then $L(\gamma) = |\gamma|$, where $|\gamma|$ is length of γ.

Definition 2. *A network in* $\mathbb{R}^n$ *is any connected complex of 1-dimensional simplexes.*

A network without vertices of degree two is called a nondegenerate network. Henceforth, we shall study only acyclic nondegenerate networks with boundary coinciding with the vertices of degree 1. Such networks are called simply networks. Any path p in a network N from a boundary point A_i *to a boundary point* A_j *denoted by* (A_iA_j)*, which is called the maximal path. A network is said to be oriented if its sides can be oriented such that every two adjacent sides are oriented oppositely to each other.*

A system of the maximal paths $\{P_1, \dots, P_m\}$ *on N is independent if there is no* $P_\alpha(\alpha \in \{1, 2, \dots, m\})$*, composed of some maximal paths from*

$$\{P_1, \dots, \hat{P}_\alpha, \dots, P_m, -P_1, \dots, -\hat{P}_\alpha, \dots, -P_m\}.$$

Definition 3. *A system of maximal paths* $\{P_j\}_{j\in J}$ *on a network N is called a basis on N if it satisfies the following conditions:*
1. *The system* $\{P_j\}_{j\in J}$ *is independent;*
2. *Every maximal path on N is a combination of paths from* $\{P_j\}_{j\in J}$.

Remarks. (See [2, 3])
1. Every network in $\mathbb{R}^n$ has exactly two orientations;
2. Every oriented network N with k boundary points has a basis consisting of $k - 1$ maximal paths;
3. For every side $a \in N$, we have

$$\mathrm{L}(a) = \int_a l(x, \vec{a}_x)ds, \quad \text{and} \quad \mathrm{L}(N) = \sum_{a\in N} \mathrm{L}(a),$$

where $\vec{a}_x$ is the unit tangent vector to the side a at $x \in a$.

Suppose that P is an arbitrary path in N consisting of the sides $a_1, \dots, a_m$. Then

$$\mathrm{P}(w) = \int_P w(x, \vec{P}_x) ds = \sum_j \int_{a_j} w(x, \vec{a}_{jx}) ds,$$

where $\vec{P}_x$ is the unit tangent vector to P at $x \in P$ and $\vec{a}_{jx}$ is the unit tangent vector to a_j at $x \in a_j$.

Theorem 1. *Let N be an oriented network with k boundary points $A_1, \dots, A_k$ in $\mathbb{R}^n$ and with a basis of maximal paths $P_1, \dots, P_{k-1}$. Suppose that there is a system of $k-1$ closed differential 1-forms $w_1, \dots, w_{k-1}$ in $\Omega^1\mathbb{R}^n$, such that*

$$\Big(\sum_{j \in J_a} \xi_j(a) w_j\Big)(x, \xi) \le l(x, \xi), \quad \forall \xi \in \mathbb{R}^n,$$

and

$$\Big(\sum_{j \in J_a} \xi_j(a) w_j\Big)(x, \vec{a}_x) = l(x, \vec{a}_x), \quad \forall x \in a,$$

where $\vec{a}_x$ is the unit tangent vector to a at $x \in a$, $J_a = \{j | a \in P_j\}$, $\xi_j(a) = 1$ if a and P_j are of the same orientation or $\xi_j(a) = -1$, if a and P_j have opposite orientations. Then N is the L-minimizing network in the class of networks with fixed topological type. (Such system $\{w_j\}_j$ is called a calibration system on $\mathbb{R}^n$ minimizing N).

Proof. Let N' be an oriented network belonging to the given topological type of N. Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is homomorphism, such that $f(N) = N'$ and $f(A_j) = A_j$, for any $j = 1, 2, \dots, k$. (The oriented on N' is induced by the oriented on N under f). Clearly, $\{P'_1 = f(P_1), \dots, P'_{k-1} = f(P_{k-1})\}$ is a basis of maximal paths in N'. We have

$$\begin{aligned}
\mathrm{L}(N) &= \sum_{a \in N} \mathrm{L}(a) \\
&= \sum_{a \in N} \int_a l(x, \vec{a}_x) ds \\
&= \sum_{a \in N} \int_a \Big(\sum_{j \in J_a} \xi_j(a) w_j\Big)(x, \vec{a}_x) ds \\
&= \sum_{j=1}^{k-1} \int_{P_j} w_j(x, \vec{P}_{jx}) ds \\
&= \sum_{j=1}^{k-1} \int_{P'_j} w_j(x', \vec{P}'_{jx}) ds'
\end{aligned}$$

$$= \sum_{a' \in N'} \int_{a'} (\sum \xi_j(a')w_j)(x', \vec{a}_{x'})ds'$$
$$\leq \sum_{a' \in N'} \int_{a'} l(x', \vec{a}'_x)ds'$$
$$= \sum_{a' \in N'} \mathrm{L}(a') = \mathrm{L}(N'),$$

where $a' = f(a), x' = f(x)$, $\vec{P}_{jx}, \vec{P}'_{jx'}$ are the unit tangent vectors to P_j and P'_j respectively at x, x' and $\vec{a}_x$, $\vec{a}'_{x'}$ are the unit tangent vectors to a and a' respectively. The theorem is proved. ∎

Now, we assume that $\vec{a} \in \mathbb{R}^n, |\vec{a}| = 1$ and l is a parallel convex lagrangian. Then, there is a differentiable 1-form $w \in \Omega^1\mathbb{R}^n$, satisfying
(1) $w(x, \vec{a}) = l(x, \vec{a}), \quad \forall x \in \mathbb{R}^n$;
(2) $w(x, \xi) \leq l(x, \xi), \quad \forall x, xi \in \mathbb{R}^n$.
Such 1-form w is called the 1-form induced by $\vec{a}$. By using Theorem 1, we obtain the following result.

Theorem 2. *Suppose that l is a convex lagrangian on $\mathbb{R}^n$. Then, every locally L-minimal oriented network consisting of straight line segment is also absolutely L-minimal in the class of the networks of a fixed topological type, with fixed boundary points.*

Proof. We shall prove the theorem by induction on the number of nodes in the network N. At first, we consider the case of N having one node B. We put

$$P_2 = (-a_1) \cup a_2,$$
$$P_3 = (-a_1) \cup a_3,$$
$$\cdots\cdots\cdots$$
$$P_k = (-a_1) \cup a_k,$$

where a_j denotes the side Ba_j, $j = 1, 2, \ldots, k$. Clearly, $\{P_2, P_3, \ldots, P_k\}$ is a basis of maximal paths for N. Let $\vec{a}_j$ be the unit tangent vector to N on a_j and w_j is the 1-form induced by $\vec{a}_j (j = 1, 2, \ldots, k)$. Then $\{w_2, w_3, \ldots, w_k\}$ is a calibration system L-minimizing N. Indeed, as we know in [4], since N is a locally L-minimal network, we have

$$\sum_{j=1}^{k} w_j = 0.$$

Now, for any $x \in N$, we consider the following cases
(1) If $x \in a_j$ then $w_j(x, \vec{a}_{jx}) = l(x, \vec{a}_{jx}), \forall j$ and $w_j(x, \xi) \leq l(x, \xi), \forall \xi$;
(2) If $x \in a_1$ then

$$\sum_{j=2}^{k} -w_j(x, \vec{a}_{jx}) = w_1(x, \vec{a}_{1x}) = l(x, \vec{a}_{1x}),$$

and

$$\sum_{j=2}^{k} -w_j(x,\xi) = w_1(x,\xi) \leq l(x,\xi), \quad \forall \xi \in \mathbb{R}^n.$$

Thus, $\{w_2, w_3, \dots, w_k\}$ satisfies the conditions of Theorem 1 and N is the L-minimizing network. Now, we consider an arbitrary locally L-minimal network N with nodes $B_1, B_2, \dots, B_p$ and the boundary points $A_1, A_2, \dots, A_k$. Further, assume that the statement of the theorem is true for the N with $p-1$ nodes $B_1, B_2, \dots, B_{p-1}$ and the boundary points $A_1, A_2, \dots, A_\alpha, B_p (\alpha < k)$. So, in N, the basis of maximal paths is

$$P_j = (A_1, A_j), \quad j = 2, 3, \dots, \alpha,$$
$$P_{\alpha+1} = (A_1, B_p).$$

There is the calibration system $\{w_2, w_3, \dots, w_\alpha, \theta_p\}$ satisfying Theorem 1 (where w_j is induced by the unit tangent vector to the side a_j crossing A_j and θ_p is induced by the unit tangent vector b_p to the side b_p crossing B_p).

In N, we choose the following basis of maximal paths:

$$\bar{P}_j = P_j, j = 2, 3, \dots, \alpha,$$
$$\bar{P}_{\alpha+1} = (A_1 A_{\alpha+1})$$
$$\dots\dots\dots\dots\dots\dots\dots.$$
$$\bar{P}_k = (A_1 A_k).$$

Suppose that $\bar{w}_{\alpha+1} \in \Omega^1\mathbb{R}^n (i = 1, 2, \dots, k-\alpha)$ is the 1-form induced by $\bar{\xi}_{\alpha+i}(a_{\alpha+i}).\vec{a}_{\alpha+i}$ (here, $\bar{\xi}_{\alpha+i}(a_{\alpha+i})$ are the sign of the side $a_{\alpha+i}$ in the path $\bar{P}_{\alpha+i}$ and $\vec{a}_{\alpha+i}$ is the unit tangent vector to the side $a_{\alpha+i}$ crossing $A_{\alpha+i}$).

We put

$$\bar{w}_j = w_j ; j = 1, 2, \dots, \alpha.$$

Then $\{w_2, w_3, \dots, w_k\}$ is the calibration system L-minimizing N. Indeed, for proving that the system $\{w_2, w_3, \dots, w_k\}$ is a calibration system, we need to check this system by the conditions of Theorem 1 for following case:

(1) The side a does not belong to any path from $\{P_{\alpha+1}, \dots, P_k\}$. We have

$$\sum_{j \in \bar{J}_a} (\bar{\xi}_j(a)\bar{w}_j)(x, \vec{a}_x) = \sum_{j \in J_a} (\xi_j(a) w_j)(x, \vec{a}_x)$$
$$= l(x, \vec{a}_x), \quad \forall x \in a,$$

and

$$\sum_{j \in \bar{J}_a} (\bar{\xi}_j(a)\bar{w}_j)(x, \xi) = \sum_{j \in J_a} (\xi_j(a) w_j)(x, \xi)$$
$$\leq l(x, \xi), \quad \forall \xi \in \mathbb{R}^n,$$

where $\bar{J}_a = \{j | a \in \bar{P}_j\}$.

(2) The side a belongs to $P_{\alpha+1}$.

In this case the side a also belongs to $\bar{P}_{\alpha+1}, \dots, \bar{P}_k$ and further, assume that the side a belongs to some paths $\bar{P}_{j_1}, \dots, \bar{P}_{j_t}$ $(2 \le j_t \le \alpha)$. We have

$$\Big(\sum_{j\in \bar{J}_a} \bar{\xi}_j(a)\bar{w}_j\Big)(x, \vec{a}_x) = \sum_{j=j_1}^{j_t} (\bar{\xi}_j(a)\bar{w}_j)(x, \vec{a}_x) + \sum_{j=\alpha+1}^{k} \bar{\xi}_j(a)\bar{w}_j(x, \vec{a}_x)$$

$$= \sum_{j=j_1}^{j_t} (\xi_j(a)w_j)(x, \vec{a}_x) + \xi_j(b_p)\theta_p(x, \vec{a}_x)$$

$$= l(x, \vec{a}_x), \quad \forall x \in a$$

and

$$\sum_{j\in \bar{J}_a} \bar{\xi}_j(a)\bar{w}_j(x, \xi) = \Big(\sum_{j=j_1}^{j_t} \xi_j(a)w_j + \xi_j(b_p)\theta_p\Big)(x, \xi)$$

$$\le l(x, \xi), \quad \forall \xi \in \mathbb{R}^n.$$

(3) The side a belongs to the set $\{a_{\alpha+1}, \dots, a_k\}$. We have

$$\bar{\xi}_j(a)\bar{w}_j(x, \vec{a}_x) = l(x, \vec{a}_x); \quad \forall x \in a$$

and

$$\bar{\xi}_j(a)\bar{w}_j(x, \xi) \le l(x, \xi); \quad \forall \xi \in \mathbb{R}^n.$$

Thus, $\{\bar{w}_2, \dots, \bar{w}_k\}$ satisfies the conditions of Theorem 1. The theorem is proved. ∎

3. L-Minimizing Networks in the Class of Networks with Fixed Boundary in $\mathbb{R}^n$

Suppose that N is an oriented network consisting of the straight segments in $\mathbb{R}^n$ with k boundary points $A_1, A_2, \dots, A_k$ and nodes $B_1, B_2, \dots, B_p$. As we know in [3], N has a finite sequence of embedded subnetworks

$$N_1 \subset N_2 \subset \dots \subset N_p = N.$$

Here, N_i has i nodes $B_1, B_2, \dots, B_i$. Let P_j be the path from A_1 to A_j, $(j = 2, \dots, k)$. Let a_j be the side ended at A_j, $\vec{a}_j$ be the unit tangent vector to a_j, and w_j be the 1-form in $\mathbb{R}^n$. Let b_i be the side with the ends B_{i-1}, B_i; $\vec{b}_i$ the unit tangent to b_i and θ the 1-form induced by $\vec{b}_i (i = 2, \dots, p)$. For every $\beta (\beta = 1, 2, \dots, p)$, we put $J_\beta = \{j|$ the side a_j crosses $B_\beta\}$ and $I_\beta = \{i|$ the side b_i crosses $B_\beta\}$.

Lemma 1. *Assume that*

$$\sum_{j\in J_\beta} w_j + \sum_{j\in I_\beta} \theta_i = 0; \quad \forall \beta = 1, 2, \dots, p.$$

Then

$$\sum_{j=2}^{k} \xi_j(a_j)w_j = \xi_1(a_1)w_1,$$

where $\xi_j(a_j)$ is the sign of a_j in P_j $(j = 1, 2, \dots, k)$.

Proof. We shall prove the theorem by induction on the number of nodes β. At first we consider the case, where N has a single node B_1. In this case $\xi_j(a_j) = -\xi_1(a_1)$, for any $j = 1, 2, \dots, k$. Further, we have $J_1 = \{1, 2, \dots, k\}, I_1 = \emptyset$. Since $\sum_{j=1}^{k} w_j = 0$, we obtain

$$\sum_{j=2}^{k} \xi_j(a_j)w_j = \xi_1(a_1)w_1.$$

Now, assume that N is a network with p nodes and N_{p-1} is a subnetwork with $p-1$ nodes satisfying the conditions of the lemma. We consider the subnetwork $\bar{N}$ consisting of B_p and the sides crossing B_p. Similarly to the proof above, we get

$$\sum_{j \in J_p, j \neq 1} \xi_j(a_j)w_j = \xi_p(b_p)\theta_p.$$

On the other hand,

$$\begin{aligned} \xi_1(a_1)w_1 &= \sum_{\beta=1}^{p-1} \sum_{j \in J_\beta, j \neq 1} \xi_j(a_j)w_j + \xi_p(b_p)\theta_p \\ &= \sum_{\beta=1}^{p-1} \sum_{j \in J_\beta, j \neq 1} \xi_j(a_j)w_j + \sum_{j \in J_p, j \neq 1} \xi_j(a_j)w_j \\ &= \sum_{j=2}^{k} \xi_j(a_j)w_j. \end{aligned}$$

The lemma is proved. ■

Corollary 1. *Assume that*

$$\sum_{j \in J_\beta} w_j + \sum_{j \in I_\beta} \theta_i = 0, \quad \forall \beta j = 1, 2, \dots, p.$$

Then

$$\int_{b_i} \theta_i(x, \vec{b}_{ix})ds = \xi_i(b_i) \sum_{j \in \alpha_i} \int_{b_i} \xi_j(a_j)w_j(x, \vec{b}_{ix})ds, \quad where\ \alpha_i = \{j | b_i \in P_j\}.$$

Proof. We consider the following subnetwork

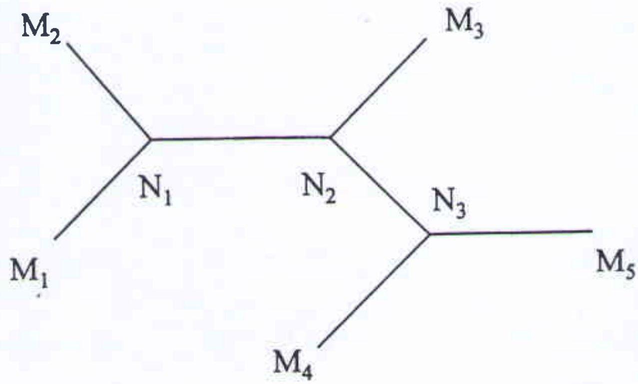
$$N' = (\cup_{j \in \alpha_i} P_j \setminus (A_1 B_i)) \cup b_i,$$

where $(A_1 B_i)$ is the path from A_1 to B_i. By using Lemma 1 for N', we obtain

$$
\begin{aligned}
L(N) &= \sum_{j=1}^{k} l(a_j) + \sum_{i=2}^{p} l(b_i) \\
&= \sum_{j=1}^{k} \int_{a_j} l(x, \vec{a}_{jx}) ds + \sum_{i=2}^{p} l(x, \vec{b}_{ix}) ds \\
&= \sum_{j=1}^{k} \int_{a_j} w_j(x, \vec{a}_{jx}) ds + \sum_{i=2}^{p} \int_{b_i} \theta_i(x, \vec{b}_{ix}) ds \\
&= \sum_{j=2}^{k} \int_{P_j} \xi_j(a) w_j \\
&= \sum_{j=2}^{k} \int_{P'_j} \xi_j(a_j) w_j \\
&= \sum_{c \in N'} \int_c \sum_{j \in J_c} \xi_j(a_j) w_j(x', \vec{c}_{x'}) ds' \\
&\leq \sum_{c \in N'} l(x', \vec{c}_{x'}) ds' \\
&= \sum_{c \in N'} L(c) = L(N'),
\end{aligned}
$$

where N' is any network having the same boundary points N, P'_j is the path in N from A_1 to A_j, c is the side of N' and $J_c = \{J | c \in P'_j\}$. The theorem is proved. ■

Example. The following network is globally minimal in $\mathbb{R}^n$ with boundary points M_1, M_2, M_3, M_4, M_5 such that the angles at the nodes N_1, N_2, N_3 equal 120° (via Euclide metric L).



Note that many papers before devoted to the investigation of globally minimal networks with only one node in considered networks.

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$$\xi_i(b_i) = \sum_{j\in\alpha_i} \xi_j(a_j)w_j.$$

Hence, we get

$$\int_{b_i} \theta_i(x,\vec{b}_{ix})ds = \xi_i(b_i)\int_{b_i} \xi_i(b_i)\theta_i(x,\vec{b}_{ix})ds$$
$$= \xi_i(b_i)\int_{b_i} \sum_{j\in\alpha_i} \xi_j(a_j)w_j(x,\vec{b}_{ix})ds. \qquad \blacksquare$$

Corollary 2. *Assume that*

$$\sum_{j\in J_\beta} w_j + \sum_{j\in J_p} \theta_i = 0, \quad \forall\beta j = 1,2,\dots,p.$$

Then

$$\sum_{j=2}^{k}\int_{P_j} \xi_j(a_j)w_j = \sum_{j=1}^{k}\int_{a_j} w_j + \sum_{j=2}^{p}\int_{b_i}\theta_i.$$

Proof. We have

$$\sum_{j=2}^{k}\int_{P_j} \xi_j(a_j)w_j = \xi_1(a_1)\int_{a_1}\sum_{j=2}^{k}\xi_j(a_1)w_j$$
$$+\sum_{j=2}^{k}\xi_j(a_j)\int_{a_j}\xi_j(a_j)w_j + \sum_{i=2}^{p}\xi_i(b_i)\int_{b_i}\sum_{j\in\alpha_i}\xi_j(a_j)w_j$$
$$= \int_{a_1} w_1 + \sum_{j=2}^{k}\int_{a_j} w_j + \sum_{i=2}^{p}\int_{b_i}\theta_i$$
$$= \sum_{j=1}^{k}\int_{a_j} w_j + \sum_{i=2}^{p}\int_{b_i}\theta_i.$$

The corollary is proved. $\blacksquare$

Theorem 3. *Suppose that l is a convex lagrangian on $\mathbb{R}^n$ and N satisfies*

$$\sum_{j\in J_\beta} w_j + \sum_{j\in I_\beta} \theta_i = 0, \quad \forall\beta j = 1,2,\dots,p,$$

and

$$\Big(\sum_{j\in T}\xi_j(a_j)w_j\Big)(x,\xi) \le l(x,\xi), \quad \forall\xi\in\mathbb{R}^n,$$

where T is any subset of $\{1,2,\dots,k\}$. Then N is the absolutely L-minimal network in the class of all the networks with fixed boundary points.

Proof. We have