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Generalized Indices of Graphs*

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Abstract. We obtain the maximum value for generalized indices of bipartite graphs of given order.

The index and period of a given digraph D are the minimum nonnegative integer $k = k(D)$ and the minimum positive integer $p = p(D)$ such that for any ordered pair of vertices x and y, there is a walk of length k from x to y if and only if there is a walk of length $k+p$ from x to y in D. A digraph D is primitive if D is strongly connected and $p(D) = 1$.

Let D be a digraph of order n with period p, and let $x \in V(D)$. The index, $k_D(x)$, of x in D is defined to be the minimum nonegative k such that for each $y \in V(D)$, there is a walk of length k from x to y if and only if there is a walk of length $k + p$ from x to y in D. If we order the vertices of D in such a way that $k_D(v_1) \leq k_D(v_2) \leq \ldots \leq k_D(v_n)$, then we call $k_D(v_i)$ the ith generalized index of D, denoted by $k(D, i)$. It is obvious that $k(D, 1) \leq \ldots \leq k(D, n) = k(D)$.

Generalized indices have been investigated in [1]. If D is primitive, then $k(D, i)$ is just the quantity $\exp_D(i)$ introduced in [2].

A symmetric digraph D is a digraph where, for any $x, y \in V(D)$, (x, y) is an arc if and only if so is (y, x). An (undirected) graph G naturally corresponds to a symmetric digraph D_G by replacing each edge $[x, y]$ by a pair of arcs (x, y) and (y, x). In this paper we will identify the graph G and the digraph D_G. Note that any edge of G corresponds to a directed cycle of length 2 in D_G. It follows that (see [1]) for any graph G, $p(G) = 1$ or 2. If G is connected, then G is primitive

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if and only if $p(G)=1$, and G is bipartite if and only if $p(G)=2$.

For a connected graph G, $d_G(x,y)$ denotes the distance between x and y for $x,y \in V(G)$.

If G is a primitive graph of order n, it is known that [3] $k(G,i) \le n-4+i$ for $3 \le i \le n$, $k(G,i) \le n-2$ for $i=1,2$ if n is even, $k(G,i) \le n-1$ for $i=1,2$ if n is odd, and that the above bound is the best possible. Hence we will be concerned with bipartite graphs. Note that it has recently been showed in [1] that $k(G,i) \le \lceil \frac{n-1}{2} \rceil + i$ for a connected bipartite graph G of order n with $1 \le i \le n$, where $\lceil a \rceil$ denotes the least integer $\ge a$. We improve this result.

Lemma 1. *Let G be a connected bipartite graph with $u \in V(G)$ and let $d = \max_{x \in V(G)} d_G(u,x)$. Then*

$$k_G(u) = d-1.$$

Proof. If there is a walk W of length $d-1$ from u to x, then there is a walk of length $d+1$ from u to x by attaching a cycle of length 2 to W.

If there is a walk of length $d+1$ from u to x , then $d_G(u,x) \le d-1$. This is because $d_G(u,x)$ and $d+1$ have the same parity and $d_G(u,x) \le d$. By attaching cycles of length 2 to the path with length $d_G(u,x)$ from u to x, we can obtain a walk of length $d-1$ from u to x.

Hence there is a walk W of length $d-1$ from u to x if and only if there is a walk of length $d+1$ from u to x, which implies that $k_G(u) \le d-1$.

On the other hand, take a vertex x such that $d = d_G(u,x)$. Clearly there is no walk of length $d-2$ from u to x, but there is a path of length d. Thus $k_G(u) \ge d-1$.

It follows that $k_G(u) = d-1$. ∎

Let $\lfloor a \rfloor$ denote the largest integer $\le a$. We have the following.

Theorem 1. *Let T be a tree of order n. Then*

$$k(T,i) \le \left\lfloor \frac{n+i-3}{2} \right\rfloor,$$

and equality holds for some i if and only if T is a path of order n.

Proof. Recall that T has either exactly one center or exactly two adjacent centers and $p(T)=2$. For a center u of T, let $d = \max_{x \in V(T)} d_T(u,x)$.

Case 1. T has exactly one center u.

In this case, T has a longest path Q with length $2d$ and center u, and $d \le \lfloor (n-1)/2 \rfloor$. By Lemma 1, $k_T(u) = d-1$. Take $x \in V(T)$.

Let $N_t(u)$ be the set of vertices reachable by a path of length t from u in T. Clearly we have $\cup_{t=0}^{d} N_t(u) = V(T)$. Let $x \in N_t(u)$. For any $y \in V(T)$, it follows from the definition of $k_T(u)$ that there is a walk of length $t + k_T(u)$ from x to y via u if and only if there is a walk of length $t + k_T(u) + 2$ from x to y via u. This implies that $k_T(x) \le t + k_T(u)$.

Note that $N_0(u) = \{u\}$, $|N_t(u)| \geq 2$ for each $1 \leq t \leq d$. For $1 \leq i \leq 2d+1$, we have

$$k(T,i) \leq \left\lfloor \frac{i}{2} \right\rfloor + k_T(u) = \left\lfloor \frac{i}{2} \right\rfloor + d - 1 \leq \left\lfloor \frac{i}{2} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor - 1 \leq \left\lfloor \frac{n+i-3}{2} \right\rfloor.$$

For $2d+1 < i \leq n$, clearly we have $k(T,i) \leq k(T, 2d+1) < \lfloor \frac{n+i-3}{2} \rfloor$.

If $k(T,i) = \lfloor (n+i-3)/2 \rfloor$ for some i, then $d = (n-1)2$, and hence T is a path of order n.

Case 2. T has exactly two adjacent centers u and v.

In this case, $d \leq \lfloor n/2 \rfloor$. By Lemma 1, we have

$$k_T(u) = d - 1 = k_T(v). \tag{1}$$

Let $N'_t(u)$ (respectively, $N'_t(v)$) be the set of vertices reachable by a path of length t from u (respectively v) in the subtree containing u (respectively, v) of $T - v$ (respectively, $T - u$). For any $x \in N'_t(u)$, $k_T(x) \leq t + k_T(u)$; and for any $x \in N'_t(v)$, $k_T(x) \leq t + k_T(v)$. In either case, from (1) we have $k_T(x) \leq t + d - 1$. Note that $|N'_t(u)| \geq 1$, $|N'_t(v)| \geq 1$ for $0 \leq t \leq d-1$. Hence, for $1 \leq i \leq 2d$ we have

$$k(T,i) \leq \left\lfloor \frac{i-1}{2} \right\rfloor + d - 1 \leq \left\lfloor \frac{i-1}{2} \right\rfloor + \left\lfloor \frac{n-2}{2} \right\rfloor \leq \left\lfloor \frac{n+i-3}{2} \right\rfloor.$$

For $i > 2d$, clearly $k(T,i) \leq k(T, 2d) < \lfloor (n+i-3)/2 \rfloor$.

If $k(T,i) = \lfloor (n+i-3)/2 \rfloor$, then $d = \frac{n}{2}$, and T is a path of order n.

Note that $k(T,i) = \lfloor (n+i-3)/2 \rfloor$ for any i if T is a path of order n. Combining Cases 1 and 2, we have $k(T,i) \leq \lfloor (n+i-3)/2 \rfloor$, and equality holds if and only if T is a path of order n. The proof is complete. ■

Theorem 2. *Let G be a bipartite graph of order n. Then*

$$k(G,i) \leq \lfloor (n+i-3)/2 \rfloor,$$

and this bound is the best possible.

Proof. First suppose that G is connected. Let T be a spanning tree of G. Let $d_1 = \max_{x \in V(G)} d_G(u,x)$, $d_2 = \max_{x \in V(T)} d_T(u,x)$. Clearly $d_1 \leq d_2$. By Lemma 1, $k_G(u) = d_1 - 1 \leq d_2 - 1 = k_T(u)$ for any $u \in V(G)$. By Theorem 1, we have the desired result.

Now suppose that G is not connected. Take any component G_1 of G. Clearly we have $k(G_1, i) \leq \lfloor (n_1 + i - 3)/2 \rfloor < \lfloor (n+i-3)/2 \rfloor$, where n_1 is the order of G_1, $1 \leq i \leq n_1 \leq n-1$. This implies that $k(G,i) < \lfloor (n+i-3)/2 \rfloor$ for $1 \leq i \leq n$. ■

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