Vietnam Journal of Mathematics **29**:1 (2001) 91-95

Vietnam Journal of MATHEMATICS

Short Communication

$\mathcal{A}$-Decomposability of the Modular Invariants of Linear Groups*

Nguyên Hưu Viêt Hưng and Trân Ngoc Nam

Department of Mathematics, Vietnam National University, 334 Nguyen Trai Str., Hanoi, Vietnam

Received July 15, 2000

1. Introduction and Statement of Results

Let $P_k := \mathbb{F}_2[x_1, \dots, x_k]$ be the polynomial algebra over the field of two elements, $\mathbb{F}_2$, in k variables $x_1, \dots, x_k$, each of degree 1. It is equipped with the usual structure of module over $GL_k := GL(k, \mathbb{F}_2)$ by means of substitutions of variables. Furthermore, the mod 2 Steenrod algebra, $\mathcal{A}$, acts upon P_k in the usual manner.

Let G be a subgroup of GL_k. Then P_k possesses the induced structure of G-module. Denote by P_k^G the subalgebra of all G-invariants in P_k. Since the action of GL_k and that of $\mathcal{A}$ on P_k commute with each other, P_k^G is also an $\mathcal{A}$-module.

In [3], the first named author is interested in the homomorphism

$$j_G : \mathbb{F}_2 \bigotimes_{\mathcal{A}} (P_k^G) \to (\mathbb{F}_2 \bigotimes_{\mathcal{A}} P_k)^G$$

induced by the identity map on P_k. He also sets up the following conjecture for $G = GL_k$ and shows that it is equivalent to a weak algebraic version of the long-standing conjecture stating that *the only spherical classes in Q_0S^0 are the elements of Hopf invariant one and those of Kervaire invariant one.*

Conjecture 1.1 ([3]). $j_{GL_k} = 0$ *in positive degrees for* $k > 2$.

This has been established for $k = 3$ in [3] and then for arbitrary $k > 2$ in

* This work was supported in part by the National Research Program, $N^0 1.4.2$.

[6]. That the conjecture is no longer valid for $k = 1$ and $k = 2$ is respectively shown in [3] to be an exposition of the existence of the Hopf invariant one and the Kervaire invariant one classes.

In the present note, we are interested in the following problem: *Which subgroup G of GL_k possesses $j_G = 0$ in positive degrees?* It should be noted that, as observed in the introduction of [3],

$$j_G = 0 \text{ in positive degrees} \quad \Longleftrightarrow \quad (P_k^G)^+ \subset \mathcal{A}^+ \cdot P_k,$$

where $(P_k^G)^+$ and $\mathcal{A}^+$ denote respectively the submodules of P_k^G and $\mathcal{A}$ consisting of all elements of positive degree. Therefore, the smaller the group G is the harder the problem turns out to be. For instance, we have understood that $j_G \neq 0$ for $G = \{1\}$, $G = GL_1$ or $G = GL_2$. Furthermore, let T_k be the Sylow 2-subgroup of GL_k consisting of all upper triangular matrices with entries 1 on the main diagonal. Then $j_{T_k} \neq 0$, indeed $V_1 = x_1$ is a T_k-invariant, however $x_1 \notin \mathcal{A}^+ \cdot P_k$.

The problem we are interested in is closely related to the *hit problem* of determination of $\mathbb{F}_2 \bigotimes_{\mathcal{A}} P_k$. This problem has first been studied by F. Peterson [11], R. Wood [15], W. Singer [14], S. Priddy [12]... who show its relationships to several classical problems in cobordism theory, modular representation theory, Adams spectral sequence for the stable homotopy of spheres, stable homotopy type of classifying spaces of finite groups. The tensor product $\mathbb{F}_2 \bigotimes_{\mathcal{A}} P_k$ has explicitly been computed for $k \leq 3$ (see [9]). It seems unlikely that an explicit description of $\mathbb{F}_2 \bigotimes_{\mathcal{A}} P_k$ for general k will appear in the near future. There is also another approach, the qualitative one, to the problem. By this we mean giving conditions on elements of P_k to show that they go to zero in $\mathbb{F}_2 \bigotimes_{\mathcal{A}} P_k$, i.e. belong to $\mathcal{A}^+ \cdot P_k$. Peterson's conjecture [11], which has been established by Wood [15], claims that $\mathbb{F}_2 \bigotimes_{\mathcal{A}} P_k = 0$ in certain degrees. Recently, Singer, Monks, Silverman... have refined Wood's method to show that many more monomials in P_k are in $\mathcal{A}^+ \cdot P_k$. (See Silverman [13] and references therein.)

The main theorem of this note shows that $j_G = 0$ in positive degrees, or equivalently $(P_k^G)^+ \subset \mathcal{A}^+ \cdot P_k$, for a family of some rather small groups G. This family contains most of the parabolic subgroups of GL_k.

Suppose G_1 is a subgroup of GL_n and G_2 is a subgroup of GL_{k-n} for $n \leq k$. Let us consider the subgroup

$$G_1 \bullet G_2 := \left\{ \begin{pmatrix} A & * \\ 0 & B \end{pmatrix} \mid A \in G_1,\, B \in G_2 \right\} \subset GL_k.$$

We are especially interested in the case $G_1 = GL_n$ and $G_2 = \mathbf{1}_{k-n}$, the unit subgroup of GL_{k-n}. Here is an interpretation of this group, which does not depend on coordinates. Let V be an $\mathbb{F}_2$-vector space of dimension k and W a vector subspace of dimension n. Then, the group $GL_n \bullet \mathbf{1}_{k-n}$ can be interpreted as the subgroup of $GL(V)$ consisting of all isomorphism $\varphi : V \to V$ with $\varphi(W) = W$ and $\tilde{\varphi} = \mathrm{id}_{V/W}$, where $\tilde{\varphi}$ denotes the induced homomorphism of φ on V/W.

We compute the algebra of $GL_n \bullet \mathbf{1}_{k-n}$-invariants by combining the works of Dickson [1] and Mui [10]. Mui's invariant of degree 2^{n-1} is defined as follows

$$V_n = \prod_{\lambda_j \in \mathbb{F}_2} (\lambda_1 x_1 + \cdots + \lambda_{n-1} x_{n-1} + x_n).$$

Dickson's invariant of degree $2^n - 2^s$ is defined by the inductive formula

$$Q_{n,s} = Q_{n-1,s-1}^2 + V_n Q_{n-1,s},$$

where, by convention, $Q_{n,n} = 1,\ _{n,s} = 0$ for $s < 0$. Then, Dickson proves in [1] that

$$\mathbb{F}_2[x_1, \dots, x_n]^{GL_n} = \mathbb{F}_2[Q_{n,0}, \dots, Q_{n,n-1}],$$

while Mui shows in [10] that

$$\mathbb{F}_2[x_1, \dots, x_k]^{T_k} = \mathbb{F}_2[V_1, \dots, V_k].$$

To generalize these works, we set

$$V_{n+1}(x_i) = \prod_{\lambda_j \in \mathbb{F}_2} (\lambda_1 x_1 + \cdots + \lambda_n x_n + x_i),$$

for $n < i \leq k$. Then, we get

Proposition 1.2. *For $k \geq n$,*

$$\mathbb{F}_2[x_1, \dots, x_k]^{GL_n \bullet \mathbf{1}_{k-n}} = \mathbb{F}_2[Q_{n,0}, \dots, Q_{n,n-1}, V_{n+1}(x_{n+1}), \dots, V_{n+1}(x_k)].$$

Theorem 1.3. (Main theorem) $j_{GL_n \bullet \mathbf{1}_{k-n}} = 0$ *in positive degrees if and only if* $n > 2$.

Obviously, $GL_3 \bullet \mathbf{1}_{k-3}$ is the smallest group among all the ones of the form $GL_n \bullet \mathbf{1}_{k-n}$ for $n > 2$. Being applied to this group, the main theorem shows that

$$\mathbb{F}_2[Q_{3,0}, Q_{3,1}, Q_{3,2}, V_4(x_4), \dots, V_4(x_k)]^+ \subset \mathcal{A}^+ \cdot P_k,$$

where $\deg Q_{3,0} = 7$, $\deg Q_{3,1} = 6$, $\deg Q_{3,2} = 4$, $\deg V_4(x_i) = 8$ for $3 < i \leq k$. This gives a large family of elements, which are hit by $\mathcal{A}$ in P_k. Remarkably, the degrees of all the generators of this polynomial algebra are small and do not depend on k.

Let us now study the parabolic subgroup of GL_k:

$$GL_{k_1,\dots,k_m} = \left\{ \begin{pmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_m \end{pmatrix} \mid A_i \in GL_{k_i} \text{ with } k_1 + \cdots + k_m = k \right\}.$$

It is easily seen that $GL_{k_1} \bullet \mathbf{1}_{k-k_1}$ is a subgroup of $GL_{k_1,\ldots,k_m}$. Therefore, we have

Corollary 1.4. *$j_{GL_{k_1,\ldots,k_m}} = 0$ in positive degrees if and only if $k_1 > 2$.*

Note that GL_k is a special case of the parabolic subgroup with $k = k_1$ and $m = 1$. Hence we obtain an alternative proof for Conjecture 1.1:

Corollary 1.5 [6]. *$j_{GL_k} = 0$ in positive degrees if and only if $k > 2$.*

The readers are referred to [4] and [5] for some problems, which are related to the main theorem and Corollary 1.5. Additionally, the problem of determination of $\mathbb{F}_2 \bigotimes_{\mathcal{A}} (P_k^{GL_k})$ and its applications have been studied by Hưng–Peterson [7, 8].

2. Outline of Proof of the Main Theorem

It suffices to show the theorem for the group $H = GL_3 \bullet \mathbf{1}_{k-3}$ for $k > 2$, since this is the smallest one of the groups $GL_n \bullet \mathbf{1}_{k-n}$ for $k \geq n > 2$.

The fundamental H-invariants $Q_{3,0}, Q_{3,1}, Q_{3,2}, V_4(x_4), \ldots, V_4(x_k)$ will respectively be denoted by $Q_0, Q_1, Q_2, W_4, \ldots, W_k$ for brevity. Using Proposition 1.2, we need only to prove that

$$(P_k^H)^+ = \mathbb{F}_2[Q_0, Q_1, Q_2, W_4, \ldots, W_k]^+ \subset \mathcal{A}^+ \cdot P_k$$

for every $k > 2$.

Definition 2.1. *Each monomial in the variables Q_0, Q_1, Q_2, $W_4, \ldots, W_k$ of P_k^H is called an H-monomial. Given an H-monomial R, let $i_0(R), i_1(R), i_2(R)$, $i_4(R), \ldots, i_k(R)$ be respectively the powers of Q_0, Q_1, Q_2, $W_4, \ldots, W_k$ in R. Set*

$$h(R) := i_0(R) + i_1(R) + i_2(R) + i_4(R) + \cdots + i_k(R).$$

Let $s(R)$ denote the minimal non-negative integer with $2^{s(R)}$ missing in the dyadic expansion of $i_2(R)$.

The following two lemmata will play a key role in the proof of the main theorem.

Lemma 2.2. *Let $R \neq 1$ be a product of some distinct elements in the set $\{Q_0, Q_1 Q_2, W_4, \ldots, W_k\}$. Then $R \in Sq^1 P_k + Sq^2 P_k$.*

Lemma 2.3. *Suppose R is an H-monomial in P_k^H, $u \neq 1$ is an arbitrary element in P_k and n is a positive integer.*

(i) *If $s(R) < n$, then $Ru^{2^n} \in \mathcal{A}^+ \cdot P_k$.*

(ii) *If $i_2(R) \equiv 2^n - 1 \,(mod\, 2^n)$ and $\binom{h(R)}{2^{n-1}} = 0$, then $Ru^{2^n} \in \mathcal{A}^+ \cdot P_k$.*

(iii) *If $i_2(R) = 2^n - 1 \geq i_1(R)$, $h(R) \equiv 2^n - 1 \,(mod\, 2^n)$ and $u \in Sq^1 P_k + Sq^2 P_k$, then $Ru^{2^n} \in \mathcal{A}^+ \cdot P_k$.*

Outline of proof of the main theorem

Suppose R is an H-monomial of positive degree in P_k^H. We need to show that $R \in \mathcal{A}^+ \cdot P_k$. Set $n := s(S)$. Then, by definition, $i_2(R) \equiv 2^n - 1 \pmod{2^{n+1}}$.

The proof proceeds by considering the following 4 cases.

Case 1: $Q_2^{2^n}$ divides R.

Case 2: There exists $u \in \{Q_0, Q_1, W_4, \ldots, W_k\}$ such that $u^{2^{n+1}}$ divides R.

Case 3: $i_0(R)$, $i_1(R)$, $i_2(R)$, $i_4(R)$, ... , $i_k(R)$ all are $\leq 2^{n+1} - 1$ and there exists $u \in \{Q_0, Q_1, Q_2, W_4, \ldots, W_k\}$ with u^{2^n} dividing R.

Case 4: $i_0(R)$, $i_1(R)$, $i_2(R)$, $i_4(R)$, ... , $i_k(R)$ all are $\leq 2^n - 1$.

The results of this note will be published in detail elsewhere.

References

1. L. E. Dickson, A fundamental system of invariants of the general modular linear group with a solution of the form problem, *Trans. Amer. Math. Soc.* **12** (1911) 75-98.
2. Nguyên H. V. Hưng, The action of the Steenrod squares on the modular invariants of linear groups, *Proc. Amer. Math. Soc.* **113** (1991) 1097-1104.
3. Nguyên H. V. Hưng, Spherical classes and the algebraic transfer, *Trans. Amer. Math. Soc.* **349** (1997) 3893-3910.
4. Nguyên H. V. Hưng, The weak conjecture on spherical classes, *Math. Zeit.* **231** (1999) 727-743.
5. Nguyên H. V. Hưng, Spherical classes and the Lambda algebra, *Trans. Amer. Math. Soc.* (to appear).
6. Nguyên H. V. Hưng and Trần Ngoc Nam, The hit problem for the Dickson algebra, *Trans. Amer. Math. Soc.* (to appear).
7. Nguyên H. V. Hưng and F. P. Peterson, $\mathcal{A}$-generators for the Dickson algebra, *Trans. Amer. Math. Soc.* **347** (1995) 4687-4728.
8. Nguyên H. V. Hưng and F. P. Peterson, Spherical classes and the Dickson algebra, *Math. Proc. Camb. Phil. Soc.* **124** (1998) 253-264.
9. M. Kameko, *Products of projective spaces as Steenrod modules*, Thesis, Johns Hopkins University, 1990.
10. H. Mui, Modular invariant theory and cohomology algebras of symmetric groups, *Jour. Fac. Sci. Univ. Tokyo* **22** (1975) 310-369.
11. F. P. Peterson, Generators of $H^*(\mathbb{RP}^\infty \wedge \mathbb{RP}^\infty)$ as a module over the Steenrod algebra, *Abstracts Amer. Math. Soc.* **833** (1987).
12. S. Priddy, On characterizing summands in the classifying space of a group, I, *Amer. Jour. Math.* **112** (1990) 737-748.
13. J. H. Silverman, Hit polynomials and the canonical antiautomorphism of the Steenrod algabra, *Proc. Amer. Math. Soc.* **123** (1995) 627-637.
14. W. M. Singer, The transfer in homological algebra, *Math. Zeit.* **202** (1989) 493-523.
15. R. M. W. Wood, Modular representations of $GL(n, \mathbb{F}_p)$ and homotopy theory, *Lecture Notes in Math.* **1172**, Springer Verlag (1985) 188-203.