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A Note on the Property of Infinitely Differentiable Functions*

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Abstract. In this paper, the existence of $\lim_{n\to\infty} \|f^{(n)}\|_{(p,q)}^{1/n}$ for an arbitrary function $f \in C^\infty(\mathbb{R})$ such that $f^{(n)} \in (L_p, l_q)$, $n = 0, 1, \ldots$ and the concrete calculation of $\lim_{n\to\infty} \|f^{(n)}\|_{(p,q)}^{1/n}$ are shown.

1. Introduction

Ha Huy Bang has given the following result [1]: Let $1 \le p \le \infty$ and $f \in C^\infty(\mathbb{R})$ such that $f^{(n)} \in L_p(\mathbb{R}), n = 0, 1, \ldots$. Then there always exists the limit

$$d_f = \lim_{n\to\infty} \|f^{(n)}\|_p^{1/n},$$

and moreover $d_f = \sigma_f = \sup\{|\xi| : \xi \in \mathrm{supp}\hat{f}\}$, where the last equality is the definition of σ_f and $\hat{f}$ is the Fourier transform of the function f. This result has been extended to any Orlicz norm by techniques special for convex functions [2].

In this paper, modifying the methods of [1, 2], we prove this result for the norm of (L_p, l_q), the amalgam of L_p and l_q on the real line $\mathbb{R}$. Note that $L_p(\mathbb{R})$ is a partial case of (L_p, l_q) spaces.

2. Result

When $1 \le p,\ q \le \infty$, denote by (L_p, l_q) the amalgam of L_p and l_q on the real

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line $\mathbb{R}$, defined by

$$(L_p, l_q) = \{f \in L_{p,loc}(\mathbb{R}) : \|f\|_{(p,q)} < \infty\},$$

where

$$\|f\|_{(p,q)} = \Big(\sum_{n=-\infty}^{\infty} \Big(\int_n^{n+1} |f(x)|^p dx \Big)^{q/p} \Big)^{1/q}$$

with the usual conventions applying when p or q is infinite. With the norm as given above, (L_p, l_q) is a Banach space [4, 6, 7]. The theory of amalgams (L_p, l_q) and their applications can be found, for example, in [3, 7].

We state our theorem:

Theorem. *Let f be in $C^\infty(\mathbb{R})$ such that $f^{(n)} \in (L_p, l_q)$ $(1 \le p, q \le \infty)$, $n = 0, 1, \ldots$. Then there always exists the limit $d_f = \lim_{n\to\infty} \|f^{(n)}\|_{(p,q)}^{1/n}$ and $d_f = \sigma_f$.*

To prove the theorem, we need the following results:

Let $\{\tau_t : t \in \mathbb{R}\}$ be the group of translations on (L_p, l_q), where τ_t is defined by $(\tau_t f)(x) = f(x - t)$.

Lemma 1. [3, 5] *Let $1 \le p, q \le \infty$. For each $t \in \mathbb{R}$, τ_t is a bounded linear operator from (L_p, l_q) to itself and $C = \sup\{\|\tau_t\| : t \in \mathbb{R}\} = \max\{2^{1/p-1/q}, 2^{1/q-1/p}\}$.*

Lemma 2. [3, 5] *Let $1 \le p, q < \infty$. If $f \in (L_p, l_q)$, then the map $t \mapsto \tau_t f$ is continuous on $\mathbb{R}$.*

By virtue of Lemma 1, we have

Lemma 3. *If $f \in (L_p, l_q)$ then $\|f(\cdot - t)\|_{(p,q)} \le C\|f\|_{(p,q)}$,* $\forall t \in \mathbb{R}$.

Proof of Theorem. We first observe that

$$\overline{\lim_{n\to\infty}} \|f^{(n)}\|_{(p,q)}^{1/n} \le \sigma_f. \tag{1}$$

It is enough to show (1) for $\sigma_f < \infty$. Given $\varepsilon > 0$, we choose a function $\varphi \in C_0^\infty(-\sigma_f - \varepsilon, \sigma_f + \varepsilon)$ such that $\varphi = 1$ in some neighborhood of $[-\sigma_f, \sigma_f]$. Hence, it follows from Bernstein's inequality for $F^{-1}\varphi$ that

$$\overline{\lim_{n\to\infty}} \|f^{(n)}\|_{(p,q)}^{1/n} \le \sigma_f + \varepsilon.$$

Letting $\varepsilon \to 0$ we get (1).

Finally, we claim that

$$\sigma_f \le \underline{\lim_{n\to\infty}} \|f^{(n)}\|_{(p,q)}^{1/n}. \tag{2}$$

Let $\psi_\lambda(x) \in C_0^\infty(\mathbb{R})$, $\psi_\lambda(x) \ge 0$, $\psi_\lambda(x) = 0$ for $|x| \ge \lambda$ and $\int_{\mathbb{R}} \psi_\lambda(x)dx = 1$. We put $f_\lambda = f * \psi_\lambda$. Then $f_\lambda \in C^\infty(\mathbb{R})$ and $f_\lambda^{(n)} = f^{(n)} * \psi_\lambda$. By virtue of Lemmas 1 - 3, we get $f_\lambda^{(n)} \in (L_p, l_q)$ and $f_\lambda^{(n)} \in L_\infty(\mathbb{R})$. It follows from [1] that

$$\sigma_{f_\lambda} = d_{f_\lambda} \leq \underline{\lim}_{n\to\infty} \|f^{(n)}\|^{1/n}_{(p,q)}.$$

Therefore, the problem is now reduced to proving that

$$|\xi| \leq \underline{\lim}_{\lambda\to 0} \sigma_{f_\lambda}, \quad \forall \xi \in \operatorname{supp}\hat{f}. \tag{3}$$

Assume that (3) is not satisfied. Then there exist a point $\xi_0 \in \operatorname{supp}\hat{f}$, a number $\varepsilon > 0$, and a subsequence λ_k (for simplicity of notation we assume that $\xi_0 > 0$) such that $\sigma_{f_{\lambda_k}} \leq \xi_0 - 2\varepsilon$, $k = 1, 2, \ldots$. If f_λ converges weakly to f then $\hat{f}_\lambda$ also converges weakly to $\hat{f}$. Therefore, if we choose a function $\varphi(x) \in C_0^\infty(\mathbb{R})$ such that $\langle \hat{f}, \varphi\rangle \neq 0$, $\operatorname{supp}\varphi(x) \subset [\xi_0 - \varepsilon, \xi_0 - \varepsilon]$, then $0 = \langle \hat{f}_k, \varphi\rangle \to \langle \hat{f}, \varphi\rangle \neq 0$, $k \to \infty$. So we arrive at a contradiction and we get (2).

To complete the proof, it remains to show that f_λ converges weakly to f.

Case 1: $1 \leq p, q < \infty$. By virtue of Lemma 2, it is clear that $\|f_\lambda - f\|_{(p,q)} \to 0$ as $\lambda \to 0$. So f_λ converges weakly to f.

Case 2: $p = \infty$ or $q = \infty$. We prove that f_λ is weakly convergent to f by contradiction: assume that for some $\varepsilon_0 > 0$, $g \in (L_{p'}, l_{q'})$ and a subsequence $\lambda_k \to 0$,

$$\left| \int_{-\infty}^{\infty} (f_{\lambda_k}(x) - f(x))g(x)dx \right| \geq \varepsilon_0, \quad k \geq 1. \tag{4}$$

Then, it is known that, $f_\lambda \to f$ as $\lambda \to 0$ in $L_{p,loc}(\mathbb{R})$. Therefore, there exists a subsequence $\{k_m\}$ (for simplicity we assume that $k_m = m$) such that $f_{\lambda_k}(x) \to f(x)$ a.e.

On the other hand, as $\{f_{\lambda_k}\}$ is a weak precompact sequence there exists a subsequence, denoted again by $\{f_{\lambda_k}\}$, and a function $f_* \in (L_p, l_q)$ such that

$$\int_{-\infty}^{\infty} f_{\lambda_k}(x)v(x)dx \to \int_{-\infty}^{\infty} f_*(x)v(x)dx, \quad \forall\, v \in (L_{p'}, l_{q'}). \tag{5}$$

Let u be an arbitrary function in $C_0^\infty(\mathbb{R})$, then $u \in (L_p, l_q)$. It follows from $f_{\lambda_k}(x) \to f(x)$ a.e. that

$$\int_{-\infty}^{\infty} f_{\lambda_k}(y)u(y)dy \to \int_{-\infty}^{\infty} f(y)u(y)dy, \quad \forall\, u \in C_0^\infty(\mathbb{R}). \tag{6}$$

Combining (5), (6) and [8, p. 15], we obtain

$$\int_{-\infty}^{\infty} f_{\lambda_k}(x)v(x)dx \to \int_{-\infty}^{\infty} f(x)v(x)dx$$

which contradicts (4). The proof of Theorem is complete. ∎

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