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# On the Mohebi-Radjabalipour Conjecture*

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**Abstract.** In 1978, S. Brown proved that each subnormal operator has an invariant subspace. In 1981, C. Apostol obtained an invariant subspace theorem on unconditionally decomposable operators. In this paper, we prove the Mohebi-Radjabalipour Conjecture under an additional condition, and obtain an invariant subspace theorem on subdecomposable operators. Our theorem contains the results of S. Brown and C. Apostol as special cases.

## 1. Introduction

In [11], Mohebi and Radjabalipour raised the following conjecture.

THE MOHEBI-RADJABALIPOUR CONJECTURE (see [11, p. 236]). Assume the operators $T \in B(X)$ and $B \in B(Z)$ on Banach spaces $X$ and $Z$, and the nonempty open set $G$ in the complex plane $\mathbb{C}$, satisfy the following conditions:

(1) $qT = Bq$ for some injective $q \in B(X, Z)$ with a closed range $qX$.

(2) There exist sequences $\{G(n)\}$ of open sets and $\{M(n)\}$ of invariant subspaces of $B$ such that $\overline{G}(n) \subset G(n+1)$, $G = \cup_n G(n)$, $\sigma(B \mid M(n)) \subset C \backslash G(n)$ and $\sigma(B/M(n)) \subset \overline{G}(n)$, $n = 1, 2, \ldots$.

(3) $\sigma(T)$ is dominating in $G$.

Then $T$ has a (non-trivial) invariant subspace.

It is easy to see that the Mohebi-Radjabalipour Conjecture, if true, will

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contain the main results of [1, 2, 4, 5, 7, 8, 10, 11] (and others) as special cases.

In the present article, using the S.Brown Technique, we prove the Mohebi–Radjabalipour Conjecture under an additional condition, and obtain an invariant subspace theorem on subdecomposable operators. Our theorem contains the main results of [2, 4] as special cases.

## 2. Preliminaries

First we recall some basic notations and facts, and give some lemmas.

We denote by $H^\infty(G)$ the Banach algebra of all bounded analytic functions on $G$ equipped with the norm $\|f\| = \sup\{|f(\lambda)|, \lambda \in G\}$. It is well known that $H^\infty(G)$ is a $w^*$-closed subspace of $L^\infty(G)$ relative to the duality $\langle L_1(G), L^\infty(G)\rangle$ and that a sequence $\{f_k\}$ in $H^\infty(G)$ converges to zero relative to $w^*$-topology if and only if it is norm-bounded and converges to zero uniformly on each compact subset of $G$. In particular, we can identify $H^\infty(G)$ with the dual space of the Banach space $Q = L_1(G)/(H^\infty(G))^\perp$. Since $Q$ is separable, the above characterization of $w^*$-convergent sequences in $H^\infty(G)$ immediately implies the $w^*$-continuity of all point evaluations $\varepsilon_\lambda : H^\infty(G) \to \mathbb{C}$, $f \to f(\lambda)$ $(\lambda \in G)$.

For $f \in H^\infty(G)$ and $\lambda \in G$ we denote by $f_\lambda$ the unique function in $H^\infty(G)$ with $(\lambda - \mu)f_\lambda(\mu) = f(\lambda) - f(\mu)$ for all $\mu \in G$. It is easy to check that for fixed $\lambda \in G$ the map $H^\infty(G) \to H^\infty(G), f \to f_\lambda$, is $w^*$-continuous.

A subset $\sigma$ of $\mathbb{C}$ will be called dominating in $G$ if $\|f\| = \sup\{|f(\lambda)|; \lambda \in \sigma \cap G\}$ holds for all $f \in H^\infty(G)$.

Let $E$ be a Banach space. Then $E^*$ denotes the dual space of $E$. If $M$ is a subspace (=closed linear manifold) of $E$, then $E/M$ denotes the quotient space of $E$ modulo $M$. If $M$ and $N$ are subspaces of $E$, then we set

$$\alpha(M, N) = \inf\{\|x - y\|; x \in M \text{ with } \|x\| = 1 \text{ and } y \in N\}.$$

If $M$ is a nonempty subset of $E$, then we denote by $M^\perp$ the annihilator of $M$ and by $\vee M$ the closed linear hull of $M$. If $N$ is a nonempty subset of $E^*$, then we denote by $N_\perp$ the preannihilator of $N$. If $E$ and $F$ are Banach spaces, then $B(E, F)$ stands for the Banach space of all continuous linear operators of $E$ into $F$. We write $B(E)$ for $B(E, E)$. For $S \in B(E)$, if $M$ is a subspace of $E$ with $SM \subset M$, then we denote by $S|M$ the restriction of $S$ onto $M$ and by $S/M$ the quotient operator induced by $S$ on $E/M$. As usual we denote by $\sigma(S)$, $\sigma_a(S)$ and $\sigma_p(S)$ the spectrum, the approximate point spectrum and the point spectrum of $S$, respectively.

**Lemma 2.1.** *Let $Y$ be a Banach space. Let $Y_0$ be a finite codimensional subspace in $Y$. If $A \in B(Y)$ and $\lambda \in \sigma_a(A)\backslash\sigma_p(A)$, then there is a sequence $\{y_n\}$ of unit vectors in $Y_0$ such that $\lim_{n\to\infty}(\lambda - A)y_n = 0$.*

*Proof.* Define $A_0 : Y_0 \to Y$ by $A_0 y = Ay$ for all $y$ in $Y_0$. Then $A_0 \in B(Y_0, Y)$. Since $\dim(Y/Y_0) < \infty$, there are a finite dimensional subspace $Y_1$ in $Y$ such that $Y_1 \cap Y_0 = \{0\}$ and $Y = Y_0 + Y_1$. Consequently

$$(\lambda - A)Y = (\lambda - A_0)Y_0 + (\lambda - A)Y_1$$

and $(\lambda - A)Y_1$ is a finite dimensional subspace in $Y$. Therefore we claim $(\lambda - A_0)Y_0$ is not closed in $Y$. In fact, if $(\lambda - A_0)Y_0$ is closed in $Y$, then by the foregoing argument $(\lambda - A)Y$ is closed in $Y$. Therefore it follows from $\lambda \notin \sigma_p(A)$ that $\lambda \notin \sigma_a(A)$, a contradiction.

Since $(\lambda - A_0)Y_0$ is not closed in $Y$, it follows that there is a sequence $\{y_n\}$ of unit vectors in $Y_0$ such that $\|y_n\| = 1$ for all $n$ and $\|(\lambda - A_0)y_n\| \to 0$, and Lemma 2.1 is proved. ■

In the rest of the present article, we shall assume that $X, Z, T, B, q, G, G(n)$ and $M(n)$ are as in the Mohebi–Radjabalipour Conjecture except Theorem 3.2, Lemma 3.3 and Corollary 3.4.

For any $z^* \in M(n)^\perp$ we define the functional $z^*/M(n) : (Z/M(n))^* \to \mathbb{C}$ by

$$\langle z^*/M(n),\ z + M(n)\rangle = \langle z^*,\ z\rangle, z \in Z.$$

It is well known that the map $\rho : z^* \longmapsto z^*/M(n), z^* \in M(n)^\perp$, is an isometric isomorphism of $M(n)^\perp$ onto $(Z/M(n))^*$.

By Lemma I.2 in [11] we have $M(n)^\perp \subset M(n+1)^\perp$ for $n = 1, 2, \cdots$. Let $M(G) = \cup_n M(n)^\perp$. Then for every $z \in Z$ and every $z^* \in M(G)$, there exists a positive integer $n$ such that $z^* \in M(n)^\perp$. Noting that $\sigma(B/M(n)) \subset \overline{G}(n) \subset G$, we can define a linear functional $z \otimes z^*$ on $H^\infty(G)$ by

$$(z \otimes z^*)(f) = \langle z^*/M(n),\ f(B/M(n))(z + M(n))\rangle, \quad f \in H^\infty(G),$$

where $f(B/M(n))$ is defined by the Riesz–Dunford functional calculus with analytic functions. It is easy to verify that $z \otimes z^*$ is a $w^*$-continuous linear functional on $H^\infty(G)$ which is independent of the particular choice of $n$.

Set $Y = qX$. Define $\widetilde{q} \colon X \to Y$ by $\widetilde{q}x = qx$, $x \in X$. Then $\widetilde{q}$ is a bounded linear operator that is bijective. Consequently the inverse operator $\widetilde{q}^{\,-1}$ is bounded, and $B|Y = \widetilde{q}T\widetilde{q}^{\,-1}$. Put $A = B|Y$. Then $A = \widetilde{q}T\widetilde{q}^{\,-1}$.

**Lemma 2.2.** *Let $n$ be a fixed positive integer. Let $y \in Y, z^* \in M(n)^\perp$ be given. Then for any $\epsilon > 0$ there exist two subspaces $Y' \subset Y, Z_0^* \in M(n)^\perp$ such that $dim(Y/Y') < \infty$, $dim(M(n)^\perp/Z_0^*) < \infty$, $Z_0^*$ is $w^*$-closed and*

$$\|y' \otimes z^*\| < \epsilon\|y'\|, \quad y' \in Y',$$

$$\|y \otimes z_0^*\| < \epsilon\|z_0^*\|, \quad z_0^* \in Z_0^*.$$

The proof of Lemma 2.2 is completely similar to that of Lemma 2.2 in [2].

## 3. Main Results

**Theorem 3.1.** *Assume the operators $T \in B(X)$ and $B \in B(Z)$ on Banach spaces $X$ and $Z$, and the nonempty open set $G$ in $\mathbb{C}$, satisfy conditions* (1), (2) *and* (3) *in the Mohebi–Radjabalipour Conjecture and the following additional condition:*

(4) *There exists a constant $a_T$, depending only on $T$, such that for each positive integer $n$, and for any finite dimensional subspace $M(\epsilon) := \vee\{y_k(\epsilon);\ y_k(\epsilon) \in Y,\ \|y_k(\epsilon)\| = 1$, and there is a $\mu_k \in \sigma(A) \cap [G\backslash\overline{G}(n)]$ with $\|(\mu_k - B)y_k(\epsilon)\| < \epsilon,\ k = 1, 2, \dots, r$, where $r$ is a positive integral$\}$, $\epsilon > 0$, as well as for any $w^*$-closed subspace $Z_0^*$ in Lemma 2.2 with $\dim(M(n)^{\perp}/Z_0^*) < \infty$, the inequality $\|y\| \le a_T\|y + z\|$ holds for all $y \in M(\epsilon), z \in (Z_0^*)^{\perp}$.*
*Then $T$ has an invariant subspace.*

*Proof.* First note that it suffices to show that $A$ has an invariant subspace, and assume without loss of generality that $\sigma(A) = \sigma_a(A)\backslash\sigma_p(A)$.

We now prove that for any given $\mu \in G$, there exist sequences $\{y_m\}_{m=0}^{\infty}$ in $Y$ and $\{z_m^*\}_{m=0}^{\infty}$ in $M(G)$ such that

$$\|y_m - y_{m-1}\| < \frac{1}{2^{m-2}},\ \|z_m^* - z_{m-1}^*\| < \frac{a_T}{2^{m-4}},\ m = 1, 2, \dots,$$
$$\|\varepsilon_\mu - y_m \otimes z_m^*\| < \frac{1}{2^{2(m-2)}},\quad m = 0, 1, 2, \dots . \tag{1}$$

Proceeding by induction, we assume that $y_j$ and $z_j^*$ have been constructed up through $j \le m$ with $y_0 = 0$ and $z_0^* = 0$. We wish to construct $y_{m+1}$ and $z_{m+1}^*$ satisfying (1).

Since $z_m^* \in M(G)$, there exists a positive integer $n(m)$ such that $z_m^* \in M(n(m))^{\perp}$. Since $\sigma(T)$ is dominating in $G$ and $A = \widetilde{q}\,T\widetilde{q}^{\,-1}$, $\sigma(A)$ is dominating in $G$. Thus by the maximum modulus principle for analytic functions, $\sigma(A) \cap [G\backslash G(n(m))]$ is clearly dominating in $G$. It follows from Proposition 2.8 in [6] (or Lemma 4.4 in [4]) that there exist $c_1, c_2, \dots, c_r \in \mathbb{C}$ and $\mu_1, \mu_2, \dots, \mu_r \in \sigma(A) \cap [G\backslash G(n(m))]$ such that

$$\sum_{k=1}^{r} |c_k| < \frac{1}{2^{2(m-1)}},\quad \|\varepsilon_\lambda - y_m \otimes z_m^* - \sum_{k=1}^{r} c_k \varepsilon_{\mu_k}\| < \frac{1}{2^{2m+1}}. \tag{2}$$

By Lemma 2.2 there exist two subspaces $Y' \subset Y$, $Z_0^* \subset M(n(m))^{\perp}$ such that $\dim(Y/Y') < \infty$, $\dim(M(n(m))^{\perp}/Z_0^*) < \infty$, $Z_0^*$ is $w^*$-closed and

$$\|y' \otimes z_m^*\| < \frac{1}{3} \cdot \frac{1}{2^{2m+1}}\|y'\|,\quad y' \in Y';$$
$$\|y_m \otimes z^*\| < \frac{1}{3a_T} \cdot \frac{1}{2^{2m+3}}\|z^*\|,\quad z^* \in Z_0^*. \tag{3}$$

Therefore $(Z_0^*)^{\perp} \supset M(n(m))$.

Fix a non-zero vector $y_0' \in Y'$. Then by Lemma III 1.1 in [12] there exists a finite codimensional subspace $Y_1$ in $Y'$ such that $\alpha(\vee\{y_0'\}, Y_1) > 1 - \frac{1}{2}$. It is plain that $Y_1$ is a finite codimensional subspace in $Y$. Fix a real number $\epsilon > 0$. Then by Lemma 2.1 there exists a vector $y_1' \in Y_1$ such that $\|y_1'\| = 1$ and $\|(\mu_1 - A)y_1'\| < \epsilon$, that is $\|(\mu_1 - B)y_1'\| < \epsilon$. Again by Lemma III 1.1 in [12] there exists a finite codimensional subspace $Y_2$ in $Y'$ such that $\alpha(\vee\{y_0', y_1'\}, Y_2) > 1 - \frac{1}{2}$. Again by Lemma 2.1 there exists $y_2' \in Y_1 \cap Y_2$ such that $\|y_2'\| = 1$ and $\|(\mu_2 - B)y_2'\| < \epsilon$. Continuing in this way we obtain vectors $y_1', y_2', \dots, y_r' \in Y'$ such that the relations

$$\|y'_k\| = 1, \|(\mu_k - B)y'_k\| < \epsilon, \alpha(\vee\{y'_0, y'_1, \cdots, y'_{k-1}\}, \vee\{y'_k, \dots, y'_r\}) > 1 - \frac{1}{2}.$$

hold for $k = 1, 2, \dots, r$. It is standard to deduce that the inequality

$$\max\{|a_k|; 1 \le k \le r\} \le 4\|\sum_{k=1}^{r} a_k y'_k\|$$

holds for all $a_1, a_2, \dots, a_r \in \mathbb{C}$ and that the canonical projection of $L = \vee_{k=0}^{r} y'_k$ onto $M = \vee_{k=1}^{r} y'_k$ has norm less or equal to 4. By the Zenger Lemma (see [3, p. 20], [9, p. 129], or [13]. If necessary, decompose $c_k$.) there exist a bounded linear functional $l$ on $L$ and complex numbers $\lambda_1, \lambda_2, \dots, \lambda_r$ such that

$$\|\sum_{k=1}^{r} \lambda_k y'_k\| \le (\sum_{k=1}^{r} |c_k|)^{1/2}, \quad \|l\| \le 4(\sum_{k=1}^{r} |c_k|)^{1/2},$$
$$\lambda_k l(y'_k) = c_k, \quad k = 1, 2, \dots, r, \quad l(y_0) = 0. \tag{4}$$

Write $\varphi_0 = l|M$, where $l|M$ denotes the restriction of $l$ onto $M$. Then $\varphi_0 \in M^*$ and

$$\|\varphi_0\| \le 4(\sum_{k=1}^{r} |c_k|)^{1/2}. \tag{5}$$

On the other hand, it follows from the condition (4) in Theorem 3.1 that $\|z_1\| \le a_\tau\|z_1 + z_2\|$ for all $z_1 \in M, z_2 \in (Z_0^*)^\perp$. Define $\varphi(z_1 + z_2) = \varphi_0(z_1)$ for all $z_1 \in M, z_2 \in (Z_0^*)^\perp$. Then by (5) $\varphi$ is a bounded linear functional on $M + (Z_0^*)^\perp$, and $\|\varphi\| \le 4a_\tau(\sum_{k=1}^{r}|c_k|)^{1/2}, \varphi((Z_0^*)^\perp) = 0$. Extend $\varphi$ to an element $v^* \in Z^*$ by the Hahn-Banach theorem. Then

$$\|v^*\| \le 4a_\tau(\sum_{k=1}^{r} |c_k|)^{1/2} \tag{6}$$

and $v^* \in Z_0^*(\subset M(n(m))^\perp)$.

Putting $v = \sum_{k=1}^{r} \lambda_k y'_k$, and noting that

$$|v^*(y'_k)f(\mu_k) - \langle v^*/M(n(m)),\ f(B/M(n(m)))(y'_k + M(n(m)))\rangle|$$
$$= |\langle v^*/M(n(m)),\ f_{\mu_k}(B/M(n(m)))(\mu_k - B/M(n(m)))(y'_k + M(n(m)))\rangle|$$
$$\le \|v^*\|\ \|f_{\mu_k}(B/M(n(m)))\|\ \|(\mu_k - B)y'_k\|$$

holds for all $f \in H^\infty(G)$, we deduce that the inequality

$$\|\sum_{k=1}^{r} c_k \varepsilon_{\mu_k} - v \otimes v^*\| < \frac{1}{3} \cdot \frac{1}{2^{2m+1}} \tag{7}$$

holds, if $\epsilon$ is small enough.

Let $y_{m+1} = y_m + v$ and $z^*_{m+1} = z^*_m + v^*$. Then by (2), (3), (4), (6) and (7) we obtain

$$\|y_{m+1} \otimes z^*_{m+1} - y_m \otimes z^*_m - \sum_{k=1}^{r} c_k \varepsilon_{\mu_k}\| \le \|v \otimes z^*_m\| + \|y_m \otimes v^*\|$$
$$+ \|v \otimes v^* - \sum_{k=1}^{r} c_k \varepsilon_{\mu_k}\| < \frac{1}{2^{2m+1}}. \tag{8}$$

Consequently, it follows from (2), (4), (6) and (8) that

$$\|y_{m+1} - y_m\| < \frac{1}{2^{m-1}}, \quad \|z^*_{m+1} - z^*_m\| < \frac{a_T}{2^{m-3}};$$

$$\|\varepsilon_\mu - y_{m+1} \otimes z^*_{m+1}\| < \frac{1}{2^{2m}}.$$

This completes the proof of (1).

Finally, we prove that $A$ has an invariant subspace. In fact, it follows from (1) that there exist $y' \in Y$, $z^* \in Z^*$ such that $y' = \lim_{m\to\infty} y_m$, $z^* = \lim_{m\to\infty} z^*_m$. Define $y = (\mu - A)y'$, where $\mu$ is as in (1). Let $Y_0$ denote the subspace generated by $(\lambda - A)^{-1}y, \lambda \notin \sigma(A) \cup \overline{G}$. As in [2, p. 10], we can prove $y \in Y_0, y' \notin Y_0$ and $AY_0 \subset Y_0$. This proves Theorem 3.1. ■

From now on assume that $B$ is an unconditionally decomposable operator on a Banach space $Z$, that $Y$ is an invariant subspace of $B$, and that $A$ denotes the restriction of $B$ onto $Y$. Then by [2] it can readily be seen that the unconditionally decomposable operator $B$ satisfies the condition (4) in Theorem 3.1, where $a_T$ is replaced by $a_B$, depending only on $B$. Thus by Theorem 3.1 and the properties of decomposable operators we obtain the following:

**Theorem 3.2.** *Let $A$ be the restriction of an unconditionally decomposable operator $B$ on a Banach spaces $Z$. Let $G$ be a nonempty open set in the complex plane $\mathbb{C}$ such that $\sigma(A)$ is dominating in $G$. Then $A$ has an invariant subspace.*

In order to derive the main results of [2, 4] from Theorem 3.2, we recall Theorem 3 in [5].

**Lemma 3.3** ([5, Theorem 3]). *Let $K$ be a compact set in the complex plane $\mathbb{C}$ with the property that for all nonempty open set $G$ in $\mathbb{C}$, the set $K$ is not dominating in $G$. Then $R(K) = C(K)$, where the symbol $C(K)$ denotes the space of all continuous functions on $K$, and $R(K)$ denotes the closure in $C(K)$ of the rational functions with poles off $K$.*

Theorem 3.2 and Lemma 3.3 together yield immediately:

**Corollary 3.4** ([2, Theorem 2.7]). *Let $A$ be the restriction of an unconditionally decomposable operator $B$ on a Banach space $Z$. Let $G$ be a simply connected set such that $R(\overline{\sigma(A) \cap G}) \neq C(\overline{\sigma(A) \cap G})$. Then $A$ has an invariant subspace.*

As in [2], the next corollary can follows from Corollary 3.4.

**Corollary 3.5.** ([4, Corollary 4.8]). *Every subnormal operator has an invariant subspace.*

*Remark.* From the above argument it is easy to see that our main results contain the main results of [2, 4] as special cases. Moreover, from the properties of decomposable operators and the counterexample II.1 in [10, p. 237] it can be seen that our main results still possesses the further value.

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