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Some Properties on Marginal Extensions and the Baer-Invariant of Groups

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Abstract. Let $\mathcal{V}$ be a variety of groups defined by the set of laws V. In this paper, we give a necessary and sufficient condition for a marginal extension to be irreducible and primitive, with respect to a given variety $\mathcal{V}$. It is also shown the existence of $\mathcal{V}$-marginal irreducible extension and a $\mathcal{V}$-covering group of a given $\mathcal{V}$-perfect group[1].

1. Introduction and Preliminaries

Let F_∞ be the free group freely generated by a countable set $X = \{x_1, x_2, \dots\}$, and V a non-empty subset of F_∞. Let $\mathcal{V}$ be a variety of groups defined by the set of laws V. There exist two important subgroups associated with a given group G and a variety $\mathcal{V}$, as follows:

$$V(G) = \langle \nu(g_1, \dots, g_r) \,|\, g_i \in G, 1 \le i \le r, \nu \in V \rangle,$$

$$V^*(G) = \{a \in G \,|\, \nu(g_1, \dots, g_i a, \dots, g_r) = \nu(g_1, \dots, g_r), g_i \in G, 1 \le i \le r, \nu \in V\},$$

which are called the *verbal subgroup* and the *marginal subgroup* of G with respect to the variety $\mathcal{V}$, respectively. See [8] for more information regarding varieties of groups.

Let $\mathcal{V}$ be a variety of groups defined by the set of laws V, and let G be a group with a free presentation

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1,$$

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where F is a free group. Then the *Baer-invariant* of G, denoted by $\mathcal{V}M(G)$, is defined to be (see also [3] or [4])

$$\frac{R\cap V(F)}{[RV^*F]},$$

where $[RV^*F]$ is the subgroup of F generated by the following set:

$$\{v(x_1,\dots,x_ir,\dots,x_s)v(x_1,\dots,x_s)^{-1}\,|\,x_i\in F, r\in R, v\in V, 1\leq i\leq n\}.$$

One can check that $[RV^*F]$ is the least normal subgroup T of F contained in R such that $R/T\subseteq V^*(F/T)$.

Note that the Baer-invariant of G is always abelian and independent of the choice of the free presentation of G (see [3]). In particular, if $\mathcal{V}$ is the variety of abelian or nilpotent groups of class at most $c\,(c\geq 1)$, then the Baer-invariant of the group G will be $(R\cap F')/[R,F]$, which by I. Schur [9] is isomorphic to the *Schur-multiplicator* of G, or $(R\cap\gamma_{c+1}(F))/[R,cF]$ (F is repeated c times), respectively (see [2,4]).

The extension $e\,:\,1\longrightarrow N\longrightarrow H\longrightarrow G\longrightarrow 1$ is said to be $\mathcal{V}$-*marginal extension* of the group N by G with respect to the variety $\mathcal{V}$, if $N\subseteq V^*(H)$. In particular, if $\mathcal{V}$ is the variety of abelian or nilpotent groups of class at most c, then the extension e is called a *central extension* or $\mathcal{N}_c$-*marginal extension*, respectively.

Clearly $e_H\,:\,1\longrightarrow V^*(H)\longrightarrow H\longrightarrow H/V^*(H)\longrightarrow 1$ is always a $\mathcal{V}$-marginal extension. Let $1\longrightarrow R\longrightarrow F\longrightarrow G\longrightarrow 1$ be a free presentation of a group G, then

$$1\longrightarrow\frac{R}{[RV^*F]}\longrightarrow\frac{F}{[RV^*F]}\longrightarrow G\longrightarrow 1$$

is a $\mathcal{V}$-marginal extension, which is called *free* $\mathcal{V}$-*marginal extension*, see [6,7] for more investigations.

Definition 1.1. *Let $\mathcal{V}$ be a variety of groups defined by the set of laws V and let $e:1\longrightarrow N\longrightarrow H\longrightarrow G\longrightarrow 1$ be a $\mathcal{V}$-marginal extension of N by a finite group G. Then the extension e is called $\mathcal{V}$-irreducible, if there is no proper subgroup K of H such that $H=NK$. If in addition $|V(H)\cap N|=|\mathcal{V}M(G)|$, then e is called a $\mathcal{V}$-primitive extension.*

We have the following useful lemma.

Lemma 1.2. *Let $1\longrightarrow R\longrightarrow F\longrightarrow G\longrightarrow 1$ be a free presentation of a group G and $1\longrightarrow A\longrightarrow B\overset{\pi}{\longrightarrow}C\longrightarrow 1$ be a $\mathcal{V}$-marginal extension of a group C. If $\alpha:G\longrightarrow C$ is a homomorphism, then there exists a homomorphism $\beta:F/[RV^*F]\longrightarrow B$ such that the following diagram is commutative*

$$\begin{array}{ccccccccc}
1\longrightarrow & \dfrac{R}{[RV^*F]} & \longrightarrow & \dfrac{F}{[RV^*F]} & \longrightarrow & G & \longrightarrow 1\\
& \downarrow\beta_| & & \downarrow\beta & & \downarrow\alpha & \\
1\longrightarrow & A & \longrightarrow & B & \overset{\pi}{\longrightarrow} & C & \longrightarrow 1,
\end{array}$$

where $\beta_|$ is the restriction of β to $R/[RV^*F]$.

Proof. Due to the freeness of F, there is a homomorphism $f : F \longrightarrow B$ such that the following diagram is commutative:

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow f & & \downarrow \alpha \\ B & \longrightarrow & C \end{array}$$

Consequently, f maps the group R into $\ker \pi$. Now if

$$w = \nu(x_1, \ldots, x_i r, \ldots, x_s)\nu(x_1, \ldots, x_s)^{-1}$$

is a generator of $[RV^*F]$, where $\nu \in V$, $r \in R$, $x_i \in F$ and $1 \le i \le s$, then we have

$$f(w) = \nu(f(x_1), \ldots, f(x_i)f(r), \ldots, f(x_s))\nu(f(x_1), \ldots, f(x_s))^{-1}.$$

Clearly, $f(r) \in A = \ker \pi$. But A is a $\mathcal{V}$-marginal subgroup of B, which implies that

$$\nu(f(x_1), \ldots, f(x_i)f(r), \ldots, f(x_s))\nu(f(x_1), \ldots, f(x_s))^{-1} = 1.$$

Thus, f maps any word of $[RV^*F]$ into 1. Consequently, $[RV^*F]$ is contained in the kernel of f. Now, f induces a homomorphism $\beta : \dfrac{F}{[RV^*F]} \longrightarrow B$ such that the following diagram is commutative

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \dfrac{R}{[RV^*F]} & \longrightarrow & \dfrac{F}{[RV^*F]} & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow \beta_| & & \downarrow \beta & & \downarrow \alpha & & \\ 1 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 1, \end{array}$$

where $\beta_|$ is the restriction of β. This completes the proof of the lemma. ■

An exact sequence $1 \to N \to H \to G \to 1$ is said to be a $\mathcal{V}$-*stem cover* of G, if $N \subseteq V(H) \bigcap V^*(H)$ and $N \cong \mathcal{V}M(G)$. Note that this isomorphism is the restriction of β to $\mathcal{V}M(G)$. In this case H is called a $\mathcal{V}$-*covering group* of G. It is clear that in finite case H is a $\mathcal{V}$-covering group of G, if and only if the exact sequence e is $\mathcal{V}$-primitive and $|H| = |G||\mathcal{V}M(G)|$.

The following lemma is also useful in our further investigations.

Lemma 1.3. *Let $\mathcal{V}$ be a variety of groups and G a group with a normal subgroup N contained in $V^*(G)$. Then the sequence*

$$\mathcal{V}M(G) \longrightarrow \mathcal{V}M(\frac{G}{N}) \longrightarrow V(G) \cap N \longrightarrow 1$$

is exact.

Proof. Assume that $G \cong F/R$, where F is a free group and $N = T/R$, for some suitable normal subgroup T of F. Since $N \subseteq V^*(G)$, then by the property of

$[TV^*F]$, we have $[TV^*F] \subseteq R$. The inclusion maps $R \cap V(F) \xrightarrow{f} T \cap V(F)$ and $T \cap V(F) \xrightarrow{g} T \cap V(F)R$ induce the sequence of homomorphisms

$$\frac{R \cap V(F)}{[RV^*F]} \xrightarrow{f^*} \frac{T \cap V(F)}{[TV^*F]} \xrightarrow{g^*} \frac{T \cap V(F)R}{R} \longrightarrow 1.$$

Owing to the modular law, $(T \cap V(F)R) = (T \cap V(F))R$, and so g^* is surjective. Clearly

$$\ker g^* = \frac{R \cap V(F)}{[TV^*F]} = Imf^*,$$

that is the above sequence is exact. Finally the domain and codomain of f^* are the groups $\mathcal{V}M(G)$ and $\mathcal{V}M(G/N)$, respectively, and

$$V(G) \cap N = (V(F)R/R) \cap (T/R) = (T \cap V(F)R)/R.$$

Thus the assertion follows. ■

Keeping the above notation we have the following

Lemma 1.4. *Let*

$$1 \longrightarrow \frac{R}{[RV^*F]} \longrightarrow \frac{F}{[RV^*F]} \longrightarrow G \longrightarrow 1$$

be a free $\mathcal{V}$-marginal extension of a finite group G, and $1 \longrightarrow N \longrightarrow H \longrightarrow G \longrightarrow 1$ be any $\mathcal{V}$-irreducible extension of G. Then H is a homomorphic image of $F/[RV^*F]$, in such a way that there exists a normal subgroup $S/[RV^*F]$ of $R/[RV^*F]$ such that

$$H \cong \frac{F/[RV^*F]}{S/[RV^*F]}, \text{ and } N \cong \frac{R/[RV^*F]}{S/[RV^*F]}.$$

Proof. By Lemma 1.2, the following diagram commutes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \frac{R}{[RV^*F]} & \longrightarrow & \frac{F}{[RV^*F]} & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow \beta_| & & \downarrow \beta & & \downarrow i & & \\ 1 & \longrightarrow & N & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 1, \end{array}$$

and so we have $N\beta(F/[RV^*F]) = H$. By the assumption H is $\mathcal{V}$-irreducible, so $\beta(F/[RV^*F]) = H$. By the commutativity of the above diagram we have $\ker \beta \subset R/[RV^*F]$. Put $S/[RV^*F] = \ker \beta$, then $H \cong (F/[RV^*F])/(S/[RV^*F])$ and so we have

$$N \cong \frac{R/[RV^*F]}{S/[RV^*F]}.$$

2. $\mathcal{V}$-Irreducible Extensions

In this section we give some properties of $\mathcal{V}$-marginal extensions and their connections with the Baer-invariants of a group. A necessary and sufficient condition is given for a $\mathcal{V}$-marginal extension to be $\mathcal{V}$-irreducible. Also the connection of such extensions with $\mathcal{V}$-perfect groups will be discussed.

Let $1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$ be a free presentation of a finite group G, where F is a finitely generated free group. Let $e : 1 \longrightarrow N \longrightarrow H \longrightarrow G \longrightarrow 1$ be a $\mathcal{V}$-irreducible extension. By Lemma 1.4, there exists a normal subgroup $\frac{S}{[RV^*F]}$ of $\frac{F}{[RV^*F]}$ such that

$$\frac{S}{[RV^*F]} \subset \frac{R}{[RV^*F]}, \quad \frac{F/[RV^*F]}{S/[RV^*F]} \cong H$$

and

$$\frac{R/[RV^*F]}{S/[RV^*F]} \cong N.$$

Clearly, if S is a normal subgroup of F such that $[RV^*F] \subset S \subset R$, then

$$1 \longrightarrow \frac{R}{S} \longrightarrow \frac{F}{S} \longrightarrow G \longrightarrow 1$$

is a $\mathcal{V}$-marginal extension of G, but it is not necessarily $\mathcal{V}$-irreducible.

In the following we give a necessary and sufficient condition for the above extension to be $\mathcal{V}$-irreducible.

Theorem 2.1 *By the above notation and assumption the $\mathcal{V}$-marginal extension $1 \longrightarrow R/S \longrightarrow F/S \longrightarrow G \longrightarrow 1$ is $\mathcal{V}$-irreducible if and only if every maximal subgroup of F, which contains $SV(F)$ also contains R.*

Proof. The sufficient condition is equivalent to saying that a maximal subgroup $T/[RV^*F]$ of $F/[RV^*F]$, with $T/[RV^*F] \supset SV(F)/[RV^*F]$, contains $R/[RV^*F]$. Clearly

$$\frac{SV(F)}{[RV^*F]} \Big/ \frac{S}{[RV^*F]}$$

is the verbal subgroup of $(F/[RV^*F])/(S/[RV^*F]) \cong H$. This is also equivalent to the fact that every maximal subgroup M of H with $M \supseteq V(H)$ contains N.

Now, assume that $1 \longrightarrow R/S \longrightarrow F/S \longrightarrow G \longrightarrow 1$ is not $\mathcal{V}$-irreducible, then there exists a proper subgroup K of H such that $KN = H$. Since $N \subseteq V^*(H)$ and by [1, Theorem 2.4] $V(H) = V(K)[NV^*H] = V(K)$ we have $V(H) \subseteq K$, but H and hence H/K is finitely generated and so H has a maximal subgroup M say, which contains K. Clearly $V(H) \subseteq M$, but M does not contain N, which gives a contradiction. This proves the "if" part of the theorem.

Conversely, if there exists a maximal subgroup M of H with $V(H) \subseteq M$ and $N \nsubseteq M$, then $H = MN$ and hence H is not $\mathcal{V}$-irreducible. ∎

Now we have the following useful lemma.

Lemma 2.2. *Let $e : 1 \longrightarrow N \longrightarrow H \xrightarrow{\phi} G \longrightarrow 1$ be a $\mathcal{V}$-primitive marginal extension of a finite group G. Then using the notation of Lemma 1.2, the map β induces the following isomorphisms:*

(i) $V(H) \cap N \cong \mathcal{V}M(G)$;

(ii) *$V(H) \cong V(F)S/S$, for some normal subgroup S of F.*

Proof.
(i) By the definition of primitivity of e and using Lemma 1.3, we conclude that $V(H) \cap N \cong \mathcal{V}M(G)$.

(ii) Let F be the free group freely generated by the set X and let $\pi : F \longrightarrow G$ be an epimorphism with $R = \ker \pi$. Clearly, for each $x \in X$ there exists $h_x \in H$ such that $\phi(h_x) = \pi(x)$. Now, put $K = \langle h_x \in H; x \in X \rangle$ we obtain $H = NK$. But as the extension e is $\mathcal{V}$-irreducible, it implies that $H = K$. Now, consider $\psi : F \longrightarrow H$ given by $\psi(x) = h_x$, for all $x \in X$. Then ψ is an epimorphism such that $\pi = \phi \circ \psi$. Clearly $\psi(R) \subseteq N$, and hence

$$\psi([RV^*F]) \subseteq [\psi(R)V^*H] \subseteq [NV^*H] = 1.$$

Thus ψ induces a homomorphism $\bar{\psi}$ from $F/[RV^*F]$ onto H, with $\ker \bar{\psi} = S/[RV^*F]$, say. This implies that

$$H \cong \frac{F}{[RV^*F]} / \frac{S}{[RV^*F]}$$

and so $V(H) \cong (V(F)S/[RV^*F])/(S/[RV^*F])$, which gives the required assertion. ■

A group G is said to be $\mathcal{V}$-*perfect* with respect to the variety $\mathcal{V}$, if $G = V(G)$. See [5] for more discussion on this concept.

Let $\mathcal{V}$ be a variety of groups, and G a $\mathcal{V}$-perfect group, then clearly every $\mathcal{V}$-covering group of G is also $\mathcal{V}$-perfect.

Now we are in a position to prove our main results.

Proposition 2.3. *Let $\mathcal{V}$ be a variety of groups, and G a $\mathcal{V}$-perfect group. If $e : 1 \longrightarrow N \longrightarrow H \longrightarrow G \longrightarrow 1$ is a $\mathcal{V}$-marginal extension, then $1 \longrightarrow N \cap V(H) \longrightarrow V(H) \longrightarrow G \longrightarrow 1$ is a $\mathcal{V}$-irreducible marginal extension.*

Proof. Since G is $\mathcal{V}$-perfect, we have

$$G \cong V(H/N) = NV(H)/N$$

and hence $NV(H) = H$. This implies that

$$\frac{V(H)}{N \cap V(H)} \cong \frac{H}{N} \cong G$$

and so $V(H)$ is an extension of G. This extension is $\mathcal{V}$-marginal, since $H = NV(H)$ implies $V^*(H) = V^*(V(H))[NV^*H]$. Hence $N \cap V(H) \subseteq V^*(V(H))$. Now if M is a subgroup of $V(H)$ such that $M(N \cap V(H)) = V(H)$, then $H = NV(H) = NM$, and since $N \subset V^*(H)$, M contains $V(H)$ and hence $M = V(H)$. This gives the irreducibility of $V(H)$.

Theorem 2.4. *Let $\mathcal{V}$ be a variety of groups, and G a $\mathcal{V}$-perfect group with a free presentation $1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$, then $V(F)/[RV^*F]$ is a $\mathcal{V}$-covering group of G.*

Proof. See [6, Theorem 3.1].

Finally, using the above theorem we have the following result

Theorem 2.5. *Let G be a finite $\mathcal{V}$-perfect group, and $e : 1 \longrightarrow N \longrightarrow H \longrightarrow G \longrightarrow 1$ be a $\mathcal{V}$-marginal extension of G. Then*
(i) *The extension e is $\mathcal{V}$-irreducible if and only if H is $\mathcal{V}$-perfect.*
(ii) *The extension e is $\mathcal{V}$-primitive if and only if H is a $\mathcal{V}$-covering group of G.*

Proof.
(i) If e is $\mathcal{V}$-irreducible, then H is $\mathcal{V}$-perfect, since $H = NV(H)$. Conversely, if $H = V(H)$ then the extension e is $\mathcal{V}$-irreducible by Proposition 2.3.
(ii) If e is $\mathcal{V}$-primitive, then it is $\mathcal{V}$-irreducible by Definition 1.1, and $H = V(H)$ by part (i). Now by Lemma 2.2, $V(H) \cap N \cong \mathcal{V}M(G)$, thus $|H| = |G||\mathcal{V}M(G)|$. Hence H is a $\mathcal{V}$-covering group of G. The converse clearly holds, since every $\mathcal{V}$-covering group of G is $\mathcal{V}$-primitive.

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