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Weighted Inequalities for Multi-Dimensional Hardy Operators

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Abstract. A characterization is found for multi-dimensional Hardy operators to be bounded from the weighted Lebesgue space $L^p(]0,\infty[^n, v(y_1,\dots,y_n)dy_1\dots dy_n)$ into $L^q(]0,\infty[^n, u(x_1,\dots,x_n)dx_1\dots dx_n)$ provided the weight $v^{1-p'}$ satisfies some doubling and reverse doubling conditions and where $1 < p \le q < \infty$, $p' = p/(p-1)$.

1. Introduction and the Results

For each integer $n \ge 1$, the n-dimensional Hardy operator H_n is defined by

$$(H_n f)(x_1,\dots,x_n) = \int_0^{x_1} \dots \int_0^{x_n} f(y_1,\dots,y_n)dy_1\dots dy_n$$

for $x_1,\dots,x_n > 0$ and all measurable functions $f(x_1,\dots,x_n) \ge 0$.

Throughout this paper it is assumed that

$$1 < p \le q < \infty, \quad p' = \frac{p}{p-1},$$

and

$u = u(x_1,\dots,x_n)$ and $v = v(y_1,\dots,y_n)$ are weights in the sense that $0 < \int_0^{R_1} \dots \int_0^{R_n} v^{1-p'}(y_1,\dots,y_n)dy_1\dots dy_n < \infty$ and $\int_{R_1}^{\infty} \dots \int_{R_n}^{\infty} u(x_1,\dots,x_n)$ $dx_1\dots dx_n < \infty$ for all $R_1,\dots,R_n > 0$.

The boundedness of H_n from the weighted Lebesgue space $L^p(v) = L^p(]0,\infty[^n, v(y_1,\dots,y_n)dy_1\dots dy_n)$ into $L^q(u) = L^q(]0,\infty[^n, u(x_1,\dots,x_n)dx_1\dots dx_n)$ is also denoted by $H_n : L^p(v) \to L^q(u)$ and means that for some constant $C > 0$

$$\left(\int_0^{\infty} \dots \int_0^{\infty} (H_n f)^q(x_1,\dots,x_n)u(x_1,\dots,x_n)dx_1\dots dx_n \right)^{1/q}$$

$$\leq C\left(\int_0^\infty \dots \int_0^\infty f^p(y_1,\dots,y_n)v(y_1,\dots,y_n)dy_1\dots dy_n\right)^{1/p} \tag{1.1}$$

for all functions $f = f(y_1,\dots,y_n) \geq 0$.

Interest on (1.1) comes from the fact that this inequality often arises in n-dimensional weighted norm inequalities for many classical operators such as strong maximal operators, Fourier and multiple Hilbert transforms [2, 3, 7].

Our purpose in this paper is to characterize weights $u = u(x_1,\dots,x_n)$ and $v = v(y_1,\dots,y_n)$ on $]0,\infty[^n$ for which (1.1) does hold and provided that $w = v^{1-p'}(y_1,\dots,y_n)$ satisfies both the doubling condition

$$\int_0^{2R_1}\dots\int_0^{2R_n} w(y_1,\dots,y_n)dy_1\dots dy_n \leq c_d \int_0^{R_1}\dots\int_0^{R_n} w(y_1,\dots,y_n)dy_1\dots dy_n \tag{1.2}$$

and the reverse doubling assumption

$$\int_0^{2^{-k_1}R_1}\dots\int_0^{2^{-k_n}R_n} w(y_1,\dots,y_n)dy_1\dots dy_n \leq c_r 2^{-(k_1a_1+\dots+k_na_n)}\int_0^{R_1}\dots\int_0^{R_n} w(y_1,\dots,y_n)dy_1\dots dy_n. \tag{1.3}$$

In (1.2) and (1.3) the nonnegative constants c_d, c_r, $a_1,\dots,a_n$ are fixed and do not depend on the arbitrary reals $R_1,\dots,R_n > 0$ and integers $k_1,\dots,k_n \geq 0$.

A characterization of (general) weights u and v for which $H_n : L^p(v) \to L^q(u)$ are well-known for $n = 1$ [1, 5] and for $n = 2$ [7]. But the problem remains unsolved for $n \geq 3$, expect when the weights are of product types in the sense that

$$u(x_1,\dots,x_n) = u_1(x_1)\times\dots\times u_n(x_n) \quad\text{and}\quad v(y_1,\dots,y_n) = v_1(y_1)\times\dots\times v_n(y_n).$$

Indeed for such weights the problem is reduced to a superposition of one-dimensional boundednesses $H_1 : L^p(v_i) \to L^q(u_i)$, $i \in \{1,\dots,n\}$. Another approach can be found in [4]. By this note we expect to bring a little contribution to the aforementioned open question, by completely solving the case when the weight $v^{1-p'}$ satisfies (1.2) and (1.3). Particularly a full solution to the boundedness $H_n : L^p(1) \to L^q(u)$ is obtained here.

In order to give an example of weights satisfying (1.2) and (1.3), let us consider an increasing function $\varphi :]0,\infty[\to]0,\infty[$ with $\varphi(.) \in \Delta_2$ in the sense that for some constant $c > 0$

$$\varphi(2t) \leq c\varphi(t) \quad \text{for all } t > 0.$$

And define

$$w(y_1,\dots,y_n) = \varphi(y_1+\dots+y_n) \quad \text{for all } y_1,\dots,y_n > 0. \tag{1.4}$$

Then such a weight w (not necessarily of product type) satisfies (1.2) and (1.3) with $a_1 = 1,\dots,a_n = 1$. This claim will be justified in the proof of the below Proposition.

A natural necessary condition for $H_n : L^p(v) \to L^q(u)$ is that for some constant $A > 0$

$$\left(\int_{R_1}^{\infty} \dots \int_{R_n}^{\infty} u(x_1, \dots, x_n)dx_1 \dots dx_n\right)^{1/q} \times \left(\int_0^{R_1} \dots \int_0^{R_n} v^{1-p'}(y_1, \dots, y_n)dy_1 \dots dy_n\right)^{1/p'} \leq A \qquad (1.5)$$

for all $R_1, \dots, R_n > 0$. For $n = 1$ (1.5) is also known, [1, 5], to be a sufficient condition for (1.1). However this is not the case for $n = 2$ as was proved in [7]. For $n \geq 2$ it can be easily proved, by applying induction arguments and Minkowski inequality, that (1.1) is equivalent to (1.5) whenever the weights are of product type.

Our main result of this paper states that condition (1.5) is equivalent to the boundedness $H_n : L^p(v) \to L^q(u)$ whenever the weight $w = v^{1-p'}$ satisfies (1.2) and (1.3).

Theorem. *Assume that $v^{1-p'}$ satisfies both the doubling condition* (1.2) *and the reverse doubling assumption* (1.3). *Then a necessary and sufficient condition for $H_n : L^p(v) \to L^q(u)$ is that for some constant $A > 0$*

$$\left(\int_{R_1}^{2R_1} \dots \int_{R_n}^{2R_n} u(x_1, \dots, x_n)dx_1 \dots dx_n\right)^{1/q} \times \left(\int_0^{R_1} \dots \int_0^{R_n} v^{1-p'}(y_1, \dots, y_n)dy_1 \dots dy_n\right)^{1/p'} \leq A \qquad (1.6)$$

for all $R_1, \dots, R_n > 0$. Precisely, the boundedness (1.1) *with some constant $C > 0$ implies condition* (1.6) *with $A = C$. And conversely,* (1.6) *with some constant A implies* (1.1) *with $C = cA$ where $c > 0$ depends only on n, p, q and the constants c_d, c_r, $a_1, \dots, a_n$ involved in assumptions* (1.2) *and* (1.3).

Since (1.1) $\Longrightarrow$ (1.5) $\Longrightarrow$ (1.6) then our task here will remain to prove the implication (1.6) $\Longrightarrow$ (1.1). Readers who are familiarized with weighted inequalities theory may observe the similarity between our above result with that for maximal operators [6] for which some A_∞ Muckenhoupt condition (see [2] for the definition) is required for $v^{1-p'}$ to get the boundedness. Both assumptions (1.2) and (1.3) are somewhat weaker than A_∞^* condition generally introduced and used to study behaviour of classical operators on product spaces [2].

The above Theorem can be used to derive boundedness results for variants of the operator H_n like

$$(H_n^* f)(x_1, \dots, x_n) = \int_{x_1}^{\infty} \dots \int_{x_n}^{\infty} f(y_1, \dots, y_n)dy_1 \dots dy_n$$

and

$$(Hf)(x_1, \dots, x_n) = \int_0^{x_1} \int_{x_2}^{\infty} \int_0^{x_3} \dots \int_0^{x_{n-1}} \int_{x_n}^{\infty} f(y_1, \dots, y_n)dy_1 \dots dy_n.$$

For instance, by duality argument, the boundedness $H_n^* : L^p(v) \to L^q(u)$ is equivalent to $H_n : L^{q'}(u^{1-q'}) \to L^{p'}(v^{1-p'})$ with $q' = q/(q-1)$. Consequently if $u(.)$ satisfies both (1.2) and (1.3) then a necessary and sufficient condition for $H_n^* : L^p(v) \to L^q(u)$ is

$$\left(\int_0^{R_1} \dots \int_0^{R_n} u(x_1, \dots, x_n) dx_1 \dots dx_n\right)^{1/q} \times \left(\int_{R_1}^{2R_1} \dots \int_{R_n}^{2R_n} v^{1-p'}(y_1, \dots, y_n) dy_1 \dots dy_n\right)^{1/p'} \leq A$$

for all $R_1, \dots, R_n > 0$. For the operator H, changes of variables lead to see that $H : L^p(v) \to L^q(u)$ is equivalent to $H : L^p(y_2^{2(p-1)} y_n^{2(p-1)} v(y_1, y_2^{-1}, y_3, \dots, y_{n-1}, y_n^{-1}))$ $\to L^q(x_2^{-2} x_n^{-2} u(x_1, x_2^{-1}, x_3, \dots, x_{n-1}, x_n^{-1}))$.

We will end with an explicit example of weights u and v (not necessarily of product types) for which (1.1) is true.

Proposition. *Let $\varphi :]0,\infty[\to]0,\infty[$ be an increasing function with $\varphi(.) \in \Delta_2$. Define the weights*

$$u(x_1, \dots, x_n) = x_1^{-(q/p'+1)} \times \dots \times x_n^{-(q/p'+1)} \varphi^{-q/p'}(x_1 + \dots + x_n)$$

and

$$v(y_1, \dots, y_n) = \varphi^{1-p}(y_1 + \dots + y_n).$$

Then the boundedness $H_n : L^p(v) \to L^q(u)$ holds.

2. Proofs of Results

Since, for the n-dimensional setting, things are often heavy to write then it is better to shorten by introducing some notations as

$$\langle \mathbf{O}, \infty \rangle =]0, \infty[^n =]0, \infty[\times \dots \times]0, \infty[,$$
$$\langle \mathbf{O}, \mathbf{R} \rangle =]0, R_1[\times \dots \times]0, R_n[,$$
$$\langle \mathbf{R}, \infty \rangle =]R_1, \infty[\times \dots \times]R_n, \infty[$$

for $\mathbf{R} = (R_1, \dots, R_n) \in \langle \mathbf{O}, \infty \rangle$,

$$\int_{\langle \mathbf{O}, \infty \rangle} g(\mathbf{x}) d\mathbf{x} = \int_0^\infty \dots \int_0^\infty g(x_1, \dots, x_n) dx_1 \dots dx_n$$

and

$$\int_{\langle \mathbf{O}, \mathbf{x} \rangle} f(\mathbf{y}) d\mathbf{y} = \int_0^{x_1} \dots \int_0^{x_n} f(y_1, \dots, y_n) dy_1 \dots dy_n = (H_n f)(\mathbf{x}).$$

With these notations the boundedness $H_n : L^p(v) \to L^q(u)$ can be written as

$$\left(\int_{\mathbf{x} \in \langle \mathbf{0}, \infty \rangle} \left[\int_{\mathbf{y} \in \langle \mathbf{0}, \mathbf{x} \rangle} f(\mathbf{y}) d\mathbf{y}\right]^q u(\mathbf{x}) d\mathbf{x}\right)^{1/q} \leq C \left(\int_{\langle \mathbf{0}, \infty \rangle} f^p(\mathbf{y}) v(\mathbf{y}) d\mathbf{y}\right)^{1/p} \quad (2.1)$$

for all functions $f = f(\mathbf{y}) \geq 0$.

Proof of Theorem

This result will immediately follow from the next two Lemmas.

Lemma 1. *If for some constants* $\widetilde{A} > 0$ *and* $\varepsilon > 1$

$$\left(\int_{\langle \mathbf{0},\mathbf{R}\rangle} v^{1-p'}(\mathbf{z})d\mathbf{z}\right)^{q/(p'\varepsilon)} \times$$

$$\int_{\mathbf{x}\in\langle \mathbf{R},\infty\rangle}\left[\int_{\mathbf{y}\in\langle \mathbf{0},\mathbf{x}\rangle}\left(\int_{\mathbf{z}\in\langle \mathbf{0},\mathbf{y}\rangle} v^{1-p'}(\mathbf{z})d\mathbf{z}\right)^{-1/\varepsilon} v^{1-p'}(\mathbf{y})d\mathbf{y}\right]^{q/p'} u(\mathbf{x})d\mathbf{x} \leq \widetilde{A}^{q} \quad (2.2)$$

for all $\mathbf{R} \in <\mathbf{0},\infty>$, *then the boundedness* $H_n : L^p(v) \to L^q(u)$ *holds with the constant* $C = \widetilde{A}$.

Proof. For $\mathbf{y}, \mathbf{x} \in <\mathbf{0},\infty>$, let us define $\varphi(\mathbf{y}) = \left(\int_{\langle \mathbf{0},\mathbf{y}\rangle} v^{1-p'}(\mathbf{z})d\mathbf{z}\right)^{(p-1)/\varepsilon}$ and $\Theta(\mathbf{x}) = \int_{\langle \mathbf{0},\mathbf{x}\rangle} \varphi^{1-p'}(\mathbf{y})v^{1-p'}(\mathbf{y})d\mathbf{y} = \int_{\mathbf{y}\in\langle \mathbf{0},\mathbf{x}\rangle}\left(\int_{\mathbf{z}\in\langle \mathbf{0},\mathbf{y}\rangle} v^{1-p'}(\mathbf{z})d\mathbf{z}\right)^{-1/\varepsilon} v^{1-p'}(\mathbf{y})d\mathbf{y}$.

With these notations, condition (2.2) is the same as

$$\varphi(\mathbf{y})\left[\int_{\langle \mathbf{y},\infty\rangle} \Theta^{\frac{q}{p'}}(\mathbf{x})u(\mathbf{x})d\mathbf{x}\right]^{p/q} \leq \widetilde{A}^{p} \quad \text{for all } \mathbf{y} \in \langle \mathbf{0},\infty\rangle.$$

And this last inequality implies $H_n : L^p(v) \to L^q(u)$ or (2.1) since

$$\left(\int_{\mathbf{x}\in\langle \mathbf{0},\infty\rangle}\left[\int_{\mathbf{y}\in\langle \mathbf{0},\mathbf{x}\rangle} f(\mathbf{y})d\mathbf{y}\right]^{q} u(\mathbf{x})d\mathbf{x}\right)^{p/q}$$

$$\leq \left(\int_{\mathbf{x}\in\langle \mathbf{0},\infty\rangle}\left[\int_{\mathbf{y}\in\langle \mathbf{0},\mathbf{x}\rangle} f^{p}(\mathbf{y})v(\mathbf{y})\varphi(\mathbf{y})d\mathbf{y}\right]^{\frac{q}{p}} \Theta^{\frac{q}{p'}}(\mathbf{x})u(\mathbf{x})d\mathbf{x}\right)^{p/q}$$

by the Hölder inequality

$$\leq \int_{\mathbf{y}\in\langle \mathbf{0},\infty\rangle} f^{p}(\mathbf{y})v(\mathbf{y}) \quad \varphi(\mathbf{y})\left[\int_{\mathbf{x}\in\langle \mathbf{y},\infty\rangle} \Theta^{q/p'}(\mathbf{x})u(\mathbf{x})d\mathbf{x}\right]^{p/q} d\mathbf{y}$$

by the Minkowski inequality since $\frac{q}{p} \geq 1$

$$\leq \widetilde{A}^{p} \int_{\langle \mathbf{0},\infty\rangle} f^{p}(\mathbf{y})v(\mathbf{y})d\mathbf{y}.$$

Lemma 2. *Suppose that assumptions* (1.2) *and* (1.3) *are satisfied with* $w = v^{1-p'}$. *Then condition* (1.6), *with some constant* $A > 0$, *implies* (2.2) *(for all* $\varepsilon > 1$*) with the constant* $\widetilde{A} = cA$*; where* c *depends on* n, p, q *and* c_r, c_d, $a_1, \ldots, a_n$ *involved in* (1.2) *and* (1.3).

Proof. This result lies on the existence of a constant $c_0 > 0$ (depending on c_d, c_r) such that

$$\int_{\mathbf{y}\in\langle\mathbf{0},\mathbf{x}\rangle}\Big(\int_{\mathbf{z}\in\langle\mathbf{0},\mathbf{y}\rangle} v^{1-p'}(\mathbf{z})d\mathbf{z}\Big)^{-1/\varepsilon} v^{1-p'}(\mathbf{y})d\mathbf{y} \leq c_0\Big(\int_{\langle\mathbf{0},\mathbf{x}\rangle} v^{1-p'}(\mathbf{y})d\mathbf{y}\Big)^{1/\varepsilon'} \tag{2.3}$$

for all $\mathbf{x} \in \langle\mathbf{0},\infty\rangle$ and where as usual $\varepsilon' = \varepsilon/(\varepsilon-1)$. The proof of (2.3) will be postponed below. For now, let us see how inequality (2.2) can be derived from condition (1.6) under assumptions (1.2) and (1.3). For simplicity of expression, the multiple sum

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\dots\sum_{k_n=0}^{\infty} \quad \text{is denoted by} \quad \sum_{\mathbf{k}=\mathbf{0}}^{\infty}.$$

Moreover let $\mathbf{1} = (1,\dots,1)$,

$2^{\mathbf{k}} = (2^{k_1},\dots,2^{k_n})$ whenever $\mathbf{k} = (k_1,\dots,k_n) \in \}-\infty,\dots,-1,0,1,\dots,\infty\{^n$

and for $\mathbf{x} = (x_1,\dots,x_n)$, $\mathbf{y} = (y_1,\dots,y_n) \in \langle\mathbf{0},\infty\rangle$ define

$$\mathbf{x}+\mathbf{y} = (x_1+y_1,\dots,x_n+y_n), \qquad \mathbf{x}\cdot\mathbf{y} = x_1y_1+\dots+x_ny_n,$$

$$\mathbf{xy} = (x_1y_1,\dots,x_ny_n), \qquad \langle\mathbf{x},\mathbf{y}\rangle =]x_1,y_1[\times\dots\times]x_n,y_n[.$$

Therefore the conclusion arises as follows

$$\begin{aligned}
&\Big(\int_{\langle\mathbf{0},\mathbf{R}\rangle} v^{1-p'}(\mathbf{z})d\mathbf{z}\Big)^{q/(p'\varepsilon)}\times\\
&\qquad\int_{\mathbf{x}\in\langle\mathbf{R},\infty\rangle}\Big[\int_{\mathbf{y}\in\langle\mathbf{0},\mathbf{x}\rangle}\Big(\int_{\mathbf{z}\in\langle\mathbf{0},\mathbf{y}\rangle} v^{1-p'}(\mathbf{z})d\mathbf{z}\Big)^{-1/\varepsilon} v^{1-p'}(\mathbf{y})d\mathbf{y}\Big]^{q/p'} u(\mathbf{x})d\mathbf{x}\\
&\leq c_1\Big(\int_{\langle\mathbf{0},\mathbf{R}\rangle} v^{1-p'}(\mathbf{z})d\mathbf{z}\Big)^{q/(p'\varepsilon)}\int_{\mathbf{x}\in\langle\mathbf{R},\infty\rangle}\Big[\int_{\mathbf{y}\in\langle\mathbf{0},\mathbf{x}\rangle} v^{1-p'}(\mathbf{y})d\mathbf{y}\Big]^{q/(p'\varepsilon')} u(\mathbf{x})d\mathbf{x} \quad \text{by (2.3)}\\
&= c_1\sum_{\mathbf{k}=\mathbf{0}}^{\infty}\Big(\int_{\langle\mathbf{0},\mathbf{R}\rangle} v^{1-p'}(\mathbf{z})d\mathbf{z}\Big)^{q/(p'\varepsilon)}\int_{\mathbf{x}\in\langle 2^{\mathbf{k}}\mathbf{R},2^{\mathbf{k}+\mathbf{1}}\mathbf{R}\rangle}\Big[\int_{\mathbf{y}\in\langle\mathbf{0},\mathbf{x}\rangle} v^{1-p'}(\mathbf{y})d\mathbf{y}\Big]^{q/(p'\varepsilon')} u(\mathbf{x})d\mathbf{x}\\
&\leq c_1\sum_{\mathbf{k}=\mathbf{0}}^{\infty}\Big(\int_{\langle 2^{\mathbf{k}}\mathbf{R},2^{\mathbf{k}+\mathbf{1}}\mathbf{R}\rangle} u(\mathbf{x})d\mathbf{x}\Big)\times\\
&\qquad\Big(\int_{\langle\mathbf{0},\mathbf{R}\rangle} v^{1-p'}(\mathbf{z})d\mathbf{z}\Big)^{q/(p'\varepsilon)}\Big(\int_{\langle\mathbf{0},2^{\mathbf{k}+\mathbf{1}}\mathbf{R}\rangle} v^{1-p'}(\mathbf{y})d\mathbf{y}\Big)^{q/(p'\varepsilon')}\\
&\leq c_2\sum_{\mathbf{k}=\mathbf{0}}^{\infty}2^{-\mathbf{k}\cdot\mathbf{a}q/(p'\varepsilon)}\Big(\int_{\langle 2^{\mathbf{k}}\mathbf{R},2^{\mathbf{k}+\mathbf{1}}\mathbf{R}\rangle} u(\mathbf{x})d\mathbf{x}\Big)\Big(\int_{\langle\mathbf{0},2^{\mathbf{k}}\mathbf{R}\rangle} v^{1-p'}(\mathbf{y})d\mathbf{y}\Big)^{q/p'}\\
&\qquad\qquad\text{by using assumptions (1.2) and (1.3)}\\
&\leq c_2A^q\sum_{\mathbf{k}=\mathbf{0}}^{\infty}2^{-\mathbf{k}\cdot\mathbf{a}q/(p'\varepsilon)} = c_3A^q \qquad \text{by condition (1.6).}
\end{aligned}$$

Finally inequality (2.3) will also follow from assumptions (1.2) and (1.3) since

$$\int_{\mathbf{y}\in\langle\mathbf{0},\mathbf{x}\rangle}\Big(\int_{\mathbf{z}\in\langle\mathbf{0},\mathbf{y}\rangle} v^{1-p'}(\mathbf{z})d\mathbf{z}\Big)^{-1/\varepsilon} v^{1-p'}(\mathbf{y})d\mathbf{y}$$
$$=\sum_{\mathbf{j}=0}^{\infty}\int_{\mathbf{y}\in\langle 2^{-(\mathbf{j}+1)}\mathbf{x},2^{-\mathbf{j}}\mathbf{x}\rangle}\Big(\int_{\mathbf{z}\in\langle\mathbf{0},\mathbf{y}\rangle} v^{1-p'}(\mathbf{z})d\mathbf{z}\Big)^{-1/\varepsilon} v^{1-p'}(\mathbf{y})d\mathbf{y}$$
$$\le\sum_{\mathbf{j}=0}^{\infty}\Big(\int_{\langle\mathbf{0},2^{-(\mathbf{j}+1)}\mathbf{x}\rangle} v^{1-p'}(\mathbf{z})d\mathbf{z}\Big)^{-1/\varepsilon}\Big(\int_{\langle 2^{-(\mathbf{j}+1)}\mathbf{x},2^{-\mathbf{j}}\mathbf{x}\rangle} v^{1-p'}(\mathbf{y})d\mathbf{y}\Big)$$
$$\le c_4\sum_{\mathbf{j}=0}^{\infty}\Big(\int_{\langle\mathbf{0},2^{-\mathbf{j}}\mathbf{x}\rangle} v^{1-p'}(\mathbf{y})d\mathbf{y}\Big)^{1/\varepsilon'}\qquad\text{by assumption (1.2)}$$
$$\le c_5\Big(\sum_{\mathbf{j}=0}^{\infty}2^{-\mathbf{j}\cdot\mathbf{a}(1/\varepsilon')}\Big)\Big(\int_{\langle\mathbf{0},\mathbf{x}\rangle} v^{1-p'}(\mathbf{y})d\mathbf{y}\Big)^{1/\varepsilon'}\qquad\text{by assumption (1.3)}$$
$$=c_6\Big(\int_{\langle\mathbf{0},\mathbf{x}\rangle} v^{1-p'}(\mathbf{y})d\mathbf{y}\Big)^{1/\varepsilon'}.\qquad\blacksquare$$

Proof of the Proposition

By the above theorem, the task is to check condition (1.6) and to see that $w = v^{1-p'}(\mathbf{y})$ satisfies both the growth assumptions (1.2) and (1.3). For notations convenience let us introduce for $\mathbf{x} = (x_1,\dots,x_n)\in\langle\mathbf{0},\infty\rangle$:

$$\overline{\mathbf{x}} = x_1+\dots+x_n,\qquad \mathbf{x}^{\mathbf{1}} = x_1\times\dots\times x_n$$

and

$$\mathbf{x}^{\gamma} = x_1^{\gamma}\times\dots\times x_n^{\gamma},\qquad \mathbf{x}^{\mathbf{a}} = x_1^{a_1}\times\dots\times x_n^{a_n}$$

whenever $\gamma\in]-\infty,\infty[$ and $\mathbf{a} = (a_1,\dots,a_n)\in\langle\mathbf{0},\infty\rangle$.

Since $\varphi(.)$ is an increasing function, then

$$\int_{\langle\mathbf{0},\mathbf{R}\rangle} v^{1-p'}(\mathbf{z})d\mathbf{z}\le\varphi(\overline{\mathbf{R}})\times\mathbf{R}^{\mathbf{1}}$$

and

$$\int_{\langle\mathbf{R},2\mathbf{R}\rangle} u(\mathbf{y})d\mathbf{y}\le\varphi^{-q/p'}(\overline{\mathbf{R}})\int_{\langle\mathbf{R},\infty\rangle}\mathbf{y}^{-(q/p'+1)}d\mathbf{y}$$
$$\le c_1\varphi^{-q/p'}(\overline{\mathbf{R}})\times(\mathbf{R}^{\mathbf{1}})^{-q/p'}$$

where $c_1>0$ depends only on n, p and q. Therefore condition (1.6) is satisfied since

$$\Big(\int_{\langle\mathbf{R},2\mathbf{R}\rangle} u(\mathbf{y})d\mathbf{y}\Big)^{1/q}\Big(\int_{\langle\mathbf{0},\mathbf{R}\rangle} v^{1-p'}(\mathbf{z})d\mathbf{z}\Big)^{1/p'}$$
$$\le c_2\Big[\varphi^{-q/p'}(\overline{\mathbf{R}})\times(\mathbf{R}^{\mathbf{1}})^{-q/p'}\Big]^{1/q}\times\Big[\varphi(\overline{\mathbf{R}})\times\mathbf{R}^{\mathbf{1}}\Big]^{1/p'} = c_2.$$

The doubling assumption (1.2) is true since, for $w = v^{1-p'}(\mathbf{y})$,

$$\int_{\langle\mathbf{0},2\mathbf{R}\rangle} w(\mathbf{y})d\mathbf{y}\le c_3\varphi(2\overline{\mathbf{R}})\times\mathbf{R}^{\mathbf{1}}\le c_4\varphi(2^{-1}\overline{\mathbf{R}})\times(2^{-1}\mathbf{R})^{\mathbf{1}}\quad\text{by }\varphi(.)\in\Delta_2$$
$$\le c_4\int_{\langle 2^{-1}\mathbf{R},\mathbf{R}\rangle}\varphi(\overline{\mathbf{y}})d\mathbf{y}\le c_4\int_{\langle\mathbf{0},\mathbf{R}\rangle} w(\mathbf{y})d\mathbf{y}.$$

And the reverse doubling condition (1.3) arises as follows

$$\begin{aligned}\int_{\langle 0,2^{-k}R\rangle} w(y)dy &\le \varphi(\overline{R}) \times (2^{-k}R)^{1} \qquad \text{because } \varphi(.) \nearrow \\ &\le c_5 2^{-k\cdot 1}\varphi(2^{-1}\overline{R}) \times (2^{-1}R)^{1} \qquad \text{since } \varphi(.) \in \triangle_2 \\ &\le c_5 2^{-k\cdot 1}\int_{\langle 0,R\rangle} w(y)dy. \end{aligned}$$ ∎

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